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SPECIAL GENERAL MEETING.

As a large number of foreign astronomers were in England in connection with the meeting of the International Astronomical Union at Cambridge, July 14–22, a Special General Meeting of the Society was held on Friday, July 24. The President, Dr. J. H. Jeans, presided, and called upon Dr. Campbell, Dr. Lindblad, Professor Shapley, Dr. H. Spencer Jones, Dr. Van Maanen, Professor Stebbins, Dr. J. S. Plaskett, and Professor de Sitter to address the meeting on the branches of research with which they were connected. A summary of the addresses is given below.

Dr. Campbell spoke about the work done at the Lick Observatory and at its southern station in Chili in measuring the radial velocity of stars down to a little below magnitude 5·5 on the visual scale. The systematic corrections to the radial velocities obtained with different instruments have been determined and applied to the individual observations in MS. form prior to publication. The resulting velocities have been made the basis for a determination of the elements of the solar motion, and some of the results have been incorporated in the Halley Lecture at Oxford this year. The velocities of nearly three thousand stars have been observed, but the maintenance of reasonable homogeneity required the rejection of several hundred. No stars fainter than 5·5 visual magnitude have been used. Two or three hundred stars were rejected because the extremely poor lines in their spectra did not permit the making of satisfactory velocity determinations. About two hundred stars were rejected because their variable velocities had not yet been investigated. Only a few stars were used from the Great Bear, the Taurus and Perseus clusters, and the Pleiades. About a dozen stars near the antapex of the solar motion were rejected because they had velocities of recession averaging more than one hundred kilometres per second, and their inclusion would have made the speed of the solar motion away from that point in the sky too great. In addition, all

ends of the spectrum. Stars of types B to A3 can be classified according to varying width of the lines of hydrogen; stars of these types with decidedly wide lines can be classified according to colour. These special classifications follow the absolute magnitudes fairly closely in groups like the Pleiades and the Perseus cluster, and they appear to be almost independent of the Harvard types when limited to stars earlier than A3.

With a prism of 6-inches aperture and a dispersion giving 1·4 mm. on the plate between H_γ and H_ϵ, stars of the 11th photographic magnitude can be reached with an exposure of from one to two hours. The distances of such stars derived by comparing the absolute with the apparent magnitudes can be utilised for the study of the distribution of stars in space; the stars of early type are especially important on account of their high absolute photographic magnitude. The distances might also be used for converting angular proper-motions into velocities—expressed, for instance, in km./sec. This has been tried for a part of the Greenwich zone between +64° and +90°. The velocity-diagrams derived in this way show clearly different effects of star-streaming. The character of the luminosity-curve for the late types might be studied by means of the cyanogen criterion. The luminosity-curve in question shows two maxima, indicating the two groups of stars commonly referred to as " giants " and " dwarfs." The observing programme for the study of spectra to be made at Upsala includes stars of known trigonometric or spectrographic parallax, stars in the Greenwich polar zone, and early-type stars in selected regions of the Milky Way to as faint a magnitude as possible. In this way it was hoped to increase materially the data bearing on the distribution of stars in space near the galactic plane.

Professor Shapley spoke about the interesting and important question as to whether obscuring nebulosity, so evident in low galactic latitudes, was also present in high galactic latitudes. Father Hagen had catalogued a large number of dark nebulæ all over the sky. More recently, Öpik at Dorpat had published an extensive investigation based on the counts of about 250,000 stars made by Miss Lukk on Paris Astrographic charts. Öpik found to his surprise that in a belt two degrees wide, extending around the sky at Dec.+24°, the sky appeared to be full of these dark markings, and he estimated that if the veil were completely removed we should see in low galactic latitudes (0° to ±19°) more than five times as many stars to mag. 14·5 as are now seen, and from 1·4 to 2·4 times as many in higher latitudes. Clearly it was a very important problem, affecting all our work on the distribution of the faint stars, clusters, and nebulæ. He had therefore taken up the matter at Harvard. The first thing was to verify the counts made by Miss Lukk by an examination of the Harvard copy of the Paris Astrographic charts, and there seemed to be no doubt of apparent vacancies. This was further verified in a general way by an examination of Harvard photographs, which showed stars of about the same limiting magnitude as the Astrographic Charts. But when counts were made on plates of longer exposure representing twenty-two selected Paris charts in various galactic latitudes, nearly all the apparent

obscurations disappeared. There were four conclusions to be drawn: (1) Hagen's dark nebulæ were apparently observable phenomena. (2) The Astrographic Charts also showed the absence of stars in certain regions. (3) The interpretation that in these regions there was real obscuring dark matter was not correct, and a different and less revolutionary one must be advanced. (4) We could continue with confidence our investigations on the distribution of the stars based on apparent and absolute magnitudes.

[This work has been published more fully in *Harvard College Observatory Circular*, No. 281.]

Dr. H. Spencer Jones then gave an account of the discovery of Nova Pictoris by Mr. Watson at Beaufort West on the morning of May 25, and of the changes in brightness and in the spectrum as derived from a preliminary examination of the plates taken at the Cape Observatory. The star reached magnitude 1·1 fifteen days after discovery, and thereafter it faded away quickly, although early in July it brightened again. On June 17, eight days after maximum, the spectrum was still a complex dark-line spectrum, but by June 24 most of the absorption had gone when the spectrum consisted of B-type absorption with A-type bright bands. The characteristic nebular lines N^1 and N^2 had not appeared up to July 1. The usual apparent shift of lines corresponding to large radial velocities was observed. [A fuller account of Nova Pictoris by Dr. Jones appears in *The Observatory* for August, p. 261.]

Dr. Van Maanen spoke on the observed internal motions in spiral nebulæ and of the possibility of systematic errors. Jackson and Smart had gone into possible sources of error in motion of the comparison stars and in asymmetrical distribution of the points measured with respect to the centre of the nebula. Their results show that the deduced internal motions were not explicable in this way. A possible error of a more serious nature is a magnitude error due to the fact that some of the comparison stars are brighter than the nebular points. This point he examined closely, and the results were expressed in a few slides shown on the screen. The first set of tabulated results gave the mean rotational components for the comparison stars of different magnitudes and for the 399 nebular points measured in M33, which were subdivided according to magnitude as well as distance from the centre. While the comparison stars show a slight increase in the rotational component with the magnitude, the table showed clearly that this effect could only account for a slight part of the rotational components of the nebular points, for which the movements are large compared with those of the comparison stars of equal magnitude. It further showed that the nebular points at the same distance from the centre show no trace of a magnitude error. Two other nebulæ, M51 and M81, show also a slight increase in the rotational components of the comparison-stars with decreasing magnitude, while the four others (M63, 94, 101, and N.G.C. 2403) show the reverse effect. The exclusion of some of the comparison stars, which have either a large rotational component or a considerable total motion, makes no difference in this respect. The conclusion to be drawn from these results is that the

rotational motion, as a whole, cannot be explained as a magnitude error.

In the case of the globular cluster M13 no rotational component was found; a fact which checks the absence of instrumental errors. It is hoped that in five or six years all the plates of the spiral nebulæ secured with the 60-inch reflector will be retaken, so that the interval of twenty years should give double the motion deduced from the ten-year interval. Another set of plates of M33 should also be taken with the 100-inch reflector in three or four years, so as to get the rotational component closer to the centre.

Professor Stebbins, after commenting on the fact that Madison Observatory was situated in the most cloudy part of the United States, where they could count on only eight or ten observing nights (of one hour at least) per month, described the photoelectric photometer used in connection with the 15-inch refractor. Dry air was forced through the apparatus to render working conditions as perfect as possible, but the weight in consequence of this and other accessories was 40 lbs. as compared with 2 lbs.—the weight of Professor Lindemann's apparatus. A new cell had been designed with all the latest improvements, but of lighter construction; the result was a piece of apparatus weighing 80 lbs.! But the new instrument was about 100 times more sensitive, and this had enabled them to register stars of the 6th and 7th magnitude. A sample of the work was then shown on a slide in which 21 Cassiopeiæ was used as the comparison star to one whose light-curve was to be determined. It was, however, found that 21 Cassiopeiæ was an eclipsing variable with a range in magnitude of four-tenths, so that the other star was used as a comparison for it! Other examples shown were the light-curves of α Orionis and α Herculis. The latter, a red star, had been observed since 1916, and at first showed a period of 120 days; but this was not supported by later observations, indicating the need for continued observations of variable red stars which often showed great irregularity.

Dr. Plaskett commenced by stating that he had the good fortune to get his results published quickly by his Government. He then went on to describe the spectroscopic work done at the Victoria Observatory. The radial velocities of some 600 stars had already been published, together with their spectroscopic parallaxes and absolute magnitudes, and also an investigation of the O-type stars. The work on spectroscopic parallaxes was proceeding, and Mr. Harper intended to continue to measure stars of A-type. Mr. H. H. Plaskett had been investigating the nebulæ, and a paper was in course of preparation on this subject. It was hoped that the work on the energy curves of stars already started by the latter would be extended to the hotter stars as a class. Personally, he was engaged in collaboration with Mr. J. A. Pearce on an investigation for radial velocity of the fainter B-type stars down to 7·5 vis. mag. and north of $-10°$. Statistics would also be forthcoming as to their spectra, distance, and absolute brightness. The work had not proceeded very far, but, as yet, the results of the radial velocities were in conflict with Dr. Campbell's results for the B-type stars. Owing to the large numbers of B8 and B9 stars the work at present was being

confined to the types Bo to B5. The mean radial velocity from 277 stars was 20·11 km./sec., and although this value would probably be slightly decreased later, owing to the presence of numerous spectroscopic binaries and to some stars being members of rapidly moving clusters, the average radial velocity would, nevertheless, be widely different from that of 7·4 km./sec. obtained by Campbell for the brighter B stars.

Professor de Sitter spoke on the work of the Leiden Observatory. He was not in the happy position of Dr. Plaskett with regard to the prompt publishing of results. There were, however, four parts of the *Annals* ready or nearly ready for the printer. Vol. xii., pt. 3, would give a first instalment of the theory of the motions of Jupiter's satellites; vol. xiii., pt. 4, would contain the reduction of the observations of 303 southern fundamental stars observed at Leiden between the years 1882 and 1897. Vol. xiv., pt. 3, would give the results of 3038 measures of double stars by Dr. van den Bos, and vol. xv., pt. 1, would give a catalogue of 1533 red stars of types K5 to Ma observed in 1921 and 1922. In the last-mentioned work the p.e. of a single observation is $\pm 0^s{\cdot}019 \sec \delta$ in R.A., and $\pm 0''{\cdot}35$ in Dec. As regards the magnitudes of the stars observed, 239 are fainter than the 9th magnitude. It might also be of interest to mention that of the 1533 stars observed, only 25 are known variables. The stars, of course, are practically all giants. The positions given in the catalogue seem to be free from magnitude error, and the systematic difference between Leiden and Greenwich and also between Leiden and Babelsberg 1920 is zero. It would naturally be concluded that the Greenwich Catalogue was also free from magnitude error. The number of proper motions available from the catalogue is 1400, and, as the probable error of one final catalogue position is $\pm 0^s{\cdot}011 \sec \delta$ in R.A. and $\pm 0''{\cdot}24$ in Dec., the p.e. of an average proper motion is $\pm 0''{\cdot}008$. There are 75 stars of which the p.m. exceeds $0^s{\cdot}0050$ or $0''{\cdot}050$, but, as the stars are not regularly distributed, these numbers have no statistical value.

In a different field of research Professor Hertzsprung has been observing in South Africa for a year and a half, and has secured 1800 plates representing 630 hours of exposure. All these plates will be passed through the microphotometer at Leiden at the rate of 400 stars per hour, and, as some of the regions include rich areas in the galaxy and magellanic clouds, it will be appreciated that there is plenty of work to be done. Many of the plates have already been examined in the blink microscope at Johannesburg, and so far 50 new variables of the Algol type have been discovered, 20 new Cepheids, and also some new variables of the RR Lyræ type. One nova has been found exhibiting a life of only one or two hours' duration. A most important feature of the work will be the statistical study of the light-curves of the Cepheid variables.

The Problem of Two Bodies with Diminishing Mass.

By W. D. MacMillan.

Introduction.

The problem of two bodies for which the masses are diminishing is one which has engaged my interest for a number of years.* The publication of a paper on this subject by Jeans in the *Monthly Notices* for 1924 November leads me to think that a somewhat different treatment may be of interest.

If x, y are the rectangular co-ordinates of one mass referred to the other

$$\ddot{x} = -k^2 \mathrm{M}\frac{x}{r^3}, \quad \ddot{y} = -k^2 \mathrm{M}\frac{y}{r^3},$$

$$\mathrm{M} = m_1 + m_2, \quad r^2 = x^2 + y^2, \quad k^2 = \text{gravitational constant} \quad . \qquad (1)$$

In polar co-ordinates

$$\ddot{r} - \frac{h^2}{r^3} + \frac{k^2 \mathrm{M}}{r^2} = 0,$$

$$r^2\dot{\theta} = h \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

in which h is the areas constant. The mass M is a function of the time, and the nature of this function must be assumed inasmuch as we know little about it, other than the fact that it is a decreasing function of the time. If we write

$$\mathrm{M} = \mathrm{M}_0/\phi(t),$$
$$r = a \,.\, \phi(t),$$

then the first equation of (2) will be homogeneous in ϕ of degree -3, provided

$$\ddot{\phi} = \frac{c}{\phi^3},$$

where c is any constant. Aside from a constant multiplier, the solution of this equation is

$$\phi = \sqrt{1 + 2\mu t + \nu^2 t^2} \qquad . \quad . \quad . \quad . \qquad (3)$$

where μ and ν are constants of integration. We have then

$$\mathrm{M} = \mathrm{M}_0/\sqrt{1 + 2\mu t + \nu^2 t^2} \qquad . \quad . \quad . \quad . \qquad (4)$$

The differential equation which this function satisfies is

$$\frac{d}{dt}\left(\frac{\mathrm{M}}{\mathrm{M}_0}\right) = -\left(\frac{\mathrm{M}}{\mathrm{M}_0}\right)^2 \sqrt{\nu^2 - (\nu^2 - \mu^2)\left(\frac{\mathrm{M}}{\mathrm{M}_0}\right)^2} \qquad . \quad . \qquad (5)$$

The rate of decrease of mass is proportional to M^2 if $\mu = \nu$, and is proportional to M^3 if $\nu = 0$.

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* See W. D. MacMillan, "On Stellar Evolution," *Astrophysical Journal*, **48**, 35 (1918); "The Structure of the Universe," *Science*, **52**, 67 (1920); "The Growth of the Solar System," *American Mathematical Monthly*, **26**, 326 (1919); "Cosmic Evolution," *Scientia*, January-February (1923).

We will now define a new independent variable by the relation

$$\tau=\int_0^t \frac{\sqrt{\nu^2-\mu^2}dt}{1+2\mu t+\nu^2t^2} \quad . \quad . \quad . \quad . \quad (6)$$

which, aside from a constant factor, is the same as the variable which Jeans denotes by the letter q. It will improve the notation to take

$$\begin{aligned} \mu&=\nu\cos\alpha, \quad \text{if } \mu<\nu, \\ \mu&=\nu\cosh\alpha, \quad \text{if } \mu>\nu \end{aligned} \quad . \quad . \quad . \quad (7)$$

We will have, then, on integrating (6),

$$\begin{aligned} &\tan\tau=\frac{\nu t\sin\alpha}{1+\nu t\cos\alpha}, \quad \nu t=\frac{\sin\tau}{\sin(\alpha-\tau)}, \quad \sqrt{1+2\mu t+\nu t^2}=\frac{\sin\alpha}{\sin(\alpha-\tau)} \quad \mu<\nu, \\ &\tanh\tau=\frac{\nu t\sinh\alpha}{1+\nu t\cosh\alpha}, \quad \nu t=\frac{\sinh\tau}{\sinh(\alpha-\tau)}, \quad \sqrt{1+2\mu t+\nu t^2}=\frac{\sinh\alpha}{\sinh(\alpha-\tau)} \quad \mu>\nu \end{aligned} \quad (8)$$

We will change also the dependent variable r, by introducing a new variable ρ, defined as follows :—

$$\begin{aligned} r&=\frac{\rho\sin\alpha}{\sin(\alpha-\tau)}, \quad \text{if } \mu<\nu, \\ r&=\frac{\rho\sinh\alpha}{\sinh(\alpha-\tau)}, \quad \text{if } \mu>\nu \end{aligned} \quad . \quad . \quad . \quad (9)$$

Let derivatives with respect to τ be denoted by accents. Then the differential equations in the new variables are

$$\begin{aligned} (\rho''+\rho)\nu^2\sin^2\alpha-\frac{h^2}{\rho^3}+\frac{k^2M_0}{\rho^2}&=0, \\ \rho^2\theta'&=\frac{h}{\nu\sin\alpha} \end{aligned} \quad . \quad . \quad . \quad . \quad (10)$$

or, if we put $\tau=\sigma\,.\,\nu\sin\alpha$,

$$\begin{aligned} \frac{d^2\rho}{d\sigma^2}+\rho\nu^2\sin^2\alpha-\frac{h}{\rho^3}+\frac{k^2M_0}{\rho^2}&=0, \\ \rho^2\frac{d\theta}{d\sigma}&=h \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Not only are equations (11) free from the independent variable explicitly, but they differ from the original equations (2) only by the addition of a force (attractive if $\sin^2\alpha>0$, repellent if $\sin^2\alpha<0$) which is proportional to the distance.

Cases for which the Transformation is Illusory.

1. If $\mu=\nu$, $\alpha=0$, and the above transformation fails. Replacing τ by $\sigma\,.\,\nu\sin\alpha$ in (8) and (9), and then passing to the limit for $\alpha=0$, we find ($\alpha=0$)

$$t=\frac{\sigma}{1-\sigma\nu}, \quad \sigma=\frac{t}{1+\nu t}, \quad \sqrt{1+2\nu t+\nu^2t^2}=\frac{1}{1-\nu\sigma}, \quad r=\frac{\rho}{1-\mu\sigma} \quad . \quad (12)$$

The differential equation (11) is unaltered except that the term in $\sin^2 a$ drops out, and these equations become identical with (2), except that the mass is constant. This is the case in which the rate of decrease of mass is proportional to the M^2.

2. If $\nu=0$, $\mu \neq 0$, then a becomes infinite and the formulæ are again indeterminate. Since $\mu=\nu \cosh a=\nu \sinh a$, we find, on passing to the limit,

$$\sqrt{1+2\mu t}=e^{\tau}, \quad \tanh \tau=\frac{\mu t}{1+\mu t}, \quad r=\rho e^{\tau}. \qquad (\nu=0,\ \mu \neq 0) \quad (13)$$

Equations (11) are unaltered except that $\nu^2 \sin^2 a$ becomes $-\mu^2$.

Solutions corresponding to Circles.

Equation (11) admits, obviously, the solution

$$\rho=a, \quad h^2=k^2 M_0 a+\nu^2 a^4 \sin^2 a, \quad \tau=l\theta, \quad l=\frac{\nu a^2 \sin a}{h} \quad . \quad (14)$$

The equations of the corresponding orbits are therefore

$$r=\frac{a \sin a}{\sin (a-l\theta)}, \quad \frac{\dot{M}}{M_0}=-\nu\left(\frac{M}{M_0}\right)^2 \sqrt{1-\left(\frac{M}{M_0}\right)^2 \sin^2 a}, \quad (\mu<\nu) \quad (15)$$

$$r=\frac{a \sinh a}{\sinh (a-l\theta)}, \quad \frac{\dot{M}}{M_0}=-\nu\left(\frac{M}{M_0}\right)^2 \sqrt{1+\left(\frac{M}{M_0}\right)^2 \sinh^2 a}, \quad (\mu>\nu) \quad (16)$$

$$r=\frac{a}{1-l\theta}, \quad l=\frac{a^2\nu}{h}, \quad \frac{\dot{M}}{M_0}=-\nu\left(\frac{M}{M_0}\right)^2, \quad (\mu=\nu) \quad . \quad . \quad . \quad (17)$$

$$r=al^{l\theta}, \quad l=\frac{a^2\mu}{h}, \quad \frac{\dot{M}}{M_0}=-\mu\left(\frac{M}{M_0}\right)^3, \quad (\nu=0,\ \mu \neq 0) \quad . \quad . \quad (18)$$

All of these curves are spirals which have asymptotes except the last. The length of the perpendicular from the origin to the asymptote is

$$p=\lim_{r=\infty} \frac{d}{d\theta}\left(\frac{1}{r}\right),$$

which, in all of the above cases, except the last, gives

$$p=\frac{h}{a\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which formula explains why the last spiral has no asymptote.

It is evident that the eccentricity of the osculating Keplerian conic increases from zero to infinity.

General Solutions of Differential Equations.

The first equation of (11) admits the integral

$$\left(\frac{d\rho}{d\sigma}\right)^2+\rho^2\nu^2 \sin^2 a+\frac{h^2}{\rho^2}-\frac{2k^2 M_0}{\rho}=c \quad . \quad . \quad . \quad . \quad (20)$$

which shows that ρ is an elliptic function of σ. Changing the independent variable from σ to θ by the relation $d\theta/d\sigma=h/\rho^2$, we have

$$\left(\frac{d\rho}{d\theta}\right)^2=\rho^2\left[-1+2\frac{k^2M_0}{h^2}\rho+\frac{c}{h^2}\rho^2-\frac{\nu^2\sin^2\alpha}{h^2}\rho^4\right] \quad . \quad . \quad (21)$$

$$h^2=k^2Ma(1-e^2)[1+\kappa^2(1-e^2)], \quad c=\frac{k^2M}{a}[-1+2\kappa^2(1+e^2)],$$

and we see that $\dfrac{d\rho}{d\theta}$ vanishes for $\rho=a(1-e)$, $\rho=a(1+e)$.

Changing from ρ to its reciprocal by the relation $\rho=a/u$, setting $\nu^2a^3\sin^2\alpha=k^2M\,.\,\kappa^2$, and then differentiating, we get

$$(1-e^2)[1+\kappa^2(1-e^2)]\left[\frac{d^2u}{d\theta^2}+u\right]=1+\frac{\kappa^2}{u^3} \quad . \quad . \quad (22)$$

For $\kappa^2=0$, this is the familiar equation of the two-body problem, the solution of which has the period 2π in θ provided $e<1$. For $\kappa^2\neq 0$ the solution is still periodic, as is shown by (11), but the period is no longer 2π. Since, for the applications which we have in mind, κ^2 is very small, it is simpler to expand the solutions of (22) in powers of κ^2 by the methods of periodic orbits. For this purpose we will define a new variable ω by the relation

$$\theta=\frac{\omega}{\sqrt{1-2\delta}} \quad . \quad . \quad . \quad . \quad (23)$$

and determine δ as a power series in κ^2 subject to the condition that the solution of (22) shall be periodic in ω with the period 2π, and shall have the initial value $u=\dfrac{1}{(1-e)}$ whatever value κ^2 may have.

Since u is a power series in κ^2, we can write

$$u=u_0+u_2\kappa^2+u_4\kappa^4+\ \ldots,$$
$$\delta=0+\delta_2\kappa^2+\delta_4\kappa^4+\ \ldots$$

Then equation (22), expanded as a power series in κ^2, is

$$\left[(1-e^2)\left(\frac{d^2u_0}{d\omega^2}+u_0\right)-1\right]\kappa^0+\left[(1-e^2)\left(\frac{d^2u_2}{d\omega^2}+u_2\right)\right.$$
$$\left.+(1-e^2)^2\left(\frac{d^2u_0}{d\omega^2}+u_0\right)-2\delta_2\frac{d^2u_0}{d\omega^2}-u_0^{-3}\right]\kappa^2+\ \ldots=0 \quad . \quad (24)$$

The solution of the terms independent of κ^2 is, of course,

$$u_0=\frac{1+e\cos\omega}{1-e^2} \quad . \quad . \quad . \quad . \quad (25)$$

which represents the ellipse. From the coefficient of κ^2 we have

$$(1-e^2)\left(\frac{d^2u_2}{d\omega^2}+u_2\right)=-(1-e^2)-2\delta_2\frac{e\cos\omega}{1-e^2}+\left(\frac{1-e^2}{1+e\cos\omega}\right)^3 \quad . \quad (26)$$

Since the free period of this differential equation is 2π and the solution is periodic with the period 2π, the right number of (26) cannot contain any terms in the first multiple of ω. That is, expanded as a Fourier series, it cannot contain a term having $\cos\omega$ as a factor. The coefficient of $\cos\omega$ arising from the last term of (26) is

$$\frac{2}{\pi}(1-e^2)^3\int_0^\pi \frac{\cos\omega\,.\,d\omega}{(1+e\cos\omega)^3} \quad . \quad . \quad . \quad (27)$$

Now

$$I=\frac{1}{\pi}\int_0^\pi \frac{dx}{a+b\cos x}=\frac{1}{\sqrt{a^2-b^2}};$$

and

$$\frac{1}{2}\frac{\partial^2 I}{\partial a\partial b}=\frac{1}{\pi}\int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^3}=\frac{-3ab}{2(a^2-b^2)^{\frac{5}{2}}}.$$

From these integrals the value of (27) is found to be $-3e\sqrt{1-e^2}$, and the entire coefficient of $\cos\omega$ in the expansion of the right member of (26) is

$$-\left(\frac{2\delta_2 e}{1-e^2}+3e\sqrt{1-e^2}\right);$$

and since this is zero, we have

$$\delta_2=-\tfrac{3}{2}(1-e^2)^{\frac{3}{2}}.$$

Equation (26) now becomes

$$\frac{d^2u_2}{d\omega^2}+u_2=-1+\frac{(1-e^2)^2}{(1+e\cos\omega)^3}, \text{ excluding terms in } \cos\omega.$$

Since u_2 vanishes for $\omega=0$, the solution as a power series in e is readily found to be

$$u_2=(1-\cos 2\omega)e^2+\tfrac{5}{16}(-\cos\omega+\cos 3\omega)e^3+\ldots.$$

Therefore the solution of (22) which satisfies all of the conditions is

$$u=\frac{1+e\cos\omega}{1-e^2}+[(1-\cos 2\omega)e^2+\tfrac{5}{16}(-\cos\omega+\cos 3\omega)e^3+\ldots]\kappa^2+\ldots,$$

and therefore

$$\frac{\rho}{a}=\frac{1-e^2}{1+e\cos\omega}+[(-1+\cos 2\omega)e^2+\tfrac{2}{16}(\cos\omega-\cos 3\omega)e^3+\ldots]\kappa^2+\ldots,$$

$$\theta=[1-\tfrac{3}{2}(1-e^2)^{\frac{3}{2}}\kappa^2+\ldots]\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

If $\kappa^2>0$, that is $\mu<\nu$, the line of apsides of the ρ, θ orbit regresses, since if $\omega=2\pi$, ρ has returned to its initial value, but $\theta=2\pi-3\pi(1-e^2)^{\frac{3}{2}}\kappa^2+\ldots$, which is less than 2π. If $\mu>\nu$, then κ^2 is negative, and the line of apsides of the ρ, θ orbit has a forward rotation.

From the areas integral, $\rho^2\theta'=h/(a^2\nu\sin\alpha)$, we find

$$\frac{h\tau}{a^2\nu\sin\alpha}=(\sqrt{1-e^2}-2e^2\kappa^2+\ldots)\theta+\text{periodic terms};$$

or, since $h=k\sqrt{M_0}\sqrt{a}\sqrt{1-e^2}\sqrt{1+\kappa^2(1-e^2)}$, $a^2\nu\sin\alpha=k\sqrt{M_0}\sqrt{a}\,.\,\kappa$,

$$\tau=\kappa\theta(1-(1+e^2\ \ldots)\kappa^2+\ \ldots)+\text{periodic terms} \quad . \quad (29)$$

The limiting value of τ is α. Hence, ignoring the periodic terms, the limiting values of θ and ρ are

$$\theta_\infty=\frac{\alpha}{\kappa}[1+(1+e^2+\ \ldots)\kappa^2+\ \ldots],$$

$$\rho_\infty=\frac{a(1-e^2)}{1+e\cos\left(\frac{\alpha}{\kappa}\right)} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Since θ has a finite limiting value these orbits also have asymptotes.

The Line of Apsides.

The apse of an orbit is given by the condition that $\dot r=0$. Now

$$\dot r=r'\dot\tau=\rho'\sin(\alpha-\tau)+\rho\cos(\alpha-\tau)=0.$$

Hence

$$\rho'=-\rho\cot(\alpha-\tau) \text{ at the apse of the } r\text{-orbit} \quad . \quad . \quad (31)$$

and therefore

$$\rho'^2+\rho^2=\rho^2/\sin^2(\alpha-\tau).$$

Hence the integral (20) becomes

$$\frac{\nu^2a^4\sin^2\alpha}{\sin^2(\alpha-\tau)}\left(\frac{\rho}{a}\right)^2=k^2Ma\left\{-1+2\kappa^2(1+e^2)+2\left(\frac{a}{\rho}\right)-(1-e^2)(1+\kappa^2(1-e^2))\left(\frac{a}{\rho}\right)^2\right\}.$$

Setting $\nu^2a^3\sin^2\alpha=k^2M\,.\,\kappa^2$, as before, and replacing ρ/a by its value in terms of ω, we find

$$\frac{\kappa^2(1-e^2)^3}{\sin^2(\alpha-\tau)}=e^2\sin^2\omega(1+e\cos\omega)^2+[1-(1+2\cos2\omega)e^2+\ldots]\kappa^2+\ldots \quad (32)$$

It is seen from (31) that ρ' is always negative at an apse of the orbit. For the perihelion point, set $\omega=2j\pi-\omega_j$ ($\omega_j>0$), and expand in powers of ω_j. It is found that

$$\omega_j=\frac{\kappa(1-e^2)^{\frac{3}{2}}}{e(1+e)}\cot(\alpha-\tau)+\ \ldots$$

and therefore

$$\theta=2j\pi-\kappa(1-e^2)^{\frac{3}{2}}\left[\frac{\cot(\alpha-\tau)}{e(1+e)}\pm\tfrac{3}{2}\tau\right]+\ \ldots \quad . \quad (33)$$

the upper sign being taken if $\mu<\nu$, the lower if $\mu>\nu$. It follows that the longitude of perihelion decreases. Similarly, it is found that at aphelion

$$\theta=(2j+1)\pi+\kappa(1-e^2)^{\frac{3}{2}}\left[\frac{\cot(\alpha-\tau)}{e(1-e)}\pm\tfrac{3}{2}\tau\right] \quad . \quad . \quad (34)$$

and therefore the longitude of aphelion increases. It seems rather curious that the two apses move in different directions, but if θ_0 denotes

the longitude of perihelion in the ordinary two-body problem, and E the eccentric anomaly, then it is easily verified that

$$\frac{da}{a}=\left(1-\frac{2a}{r}\right)\frac{d\mathrm{M}}{\mathrm{M}},\quad de=-\frac{a(1-e^2)}{r}\cos\mathrm{E}\,.\,\frac{d\mathrm{M}}{\mathrm{M}},\quad d\theta_0=-\frac{a\sqrt{1-e^2}}{re}\,.\sin\mathrm{E}\,.\,\frac{d\mathrm{M}}{\mathrm{M}} \qquad (35)$$

From the last of these formulæ it is seen that the longitude of the perihelion of the orbit increases from perihelion to aphelion, and decreases from aphelion to perihelion, for a decreasing mass. Thus the line of apsides oscillates, and the amplitude of the oscillations increases with the time.

Changes in Major Axis and Eccentricity.

Let A be the semi-axis major and ϵ the eccentricity of the osculating Keplerian orbit. Then

$$\dot{r}^2+\frac{h^2}{r^2}-\frac{2k^2\mathrm{M}}{r^2}=-\frac{k^2\mathrm{M}}{\mathrm{A}},\quad h^2=k^2\mathrm{MA}(1-\epsilon^2) \qquad . \quad . \quad (36)$$

Now

$$\dot{r}=\nu[\rho'\sin(\alpha-\tau)+\rho\cos(\alpha-\tau)],$$

and if we restrict ourselves to points on the orbit for which $\rho'=0$, we will have, at such points,

$$\dot{r}=\nu\rho\cos(\alpha-\tau),\quad r=\frac{\rho\sin\alpha}{\sin(\alpha-\tau)},$$

with ρ constant. If for simplicity of notation we set

$$s=\frac{\sin(\alpha-\tau)}{\sin\alpha},\quad (1>s\geq 0),\quad \text{so that } \mathrm{M}=\mathrm{M}_0 s \qquad . \quad . \quad (37)$$

then (36) becomes

$$\nu^2\rho^2\cos^2(\alpha-\tau)+\left(\frac{h^2}{\rho^2}-\frac{2k^2\mathrm{M}_0}{\rho}\right)s^2=-\frac{k^2\mathrm{M}_0}{\mathrm{A}}s \qquad . \quad . \quad (38)$$

If τ vanishes at an apse of the ρ-orbit, then at $\tau=0$ (38) becomes

$$\nu^2\rho^2\cos^2\alpha+\left(\frac{h^2}{\rho^2}-\frac{2k^2\mathrm{M}_0}{\rho}\right)=-\frac{k^2\mathrm{M}_0}{\mathrm{A}_0} \qquad . \quad . \quad (39)$$

On multiplying (39) by s^2 and subtracting from (38), we have

$$\nu^2\rho^2[\cos^2(\alpha-\tau)-s^2\cos^2\alpha]=\frac{k^2\mathrm{M}_0}{\mathrm{A}_0}s^2-\frac{k^2\mathrm{M}_0}{\mathrm{A}}s,$$

or

$$\nu^2\rho^2(1-s^2)=k^2\mathrm{M}_0\left(\frac{s^2}{\mathrm{A}_0}-\frac{s}{\mathrm{A}}\right) \qquad . \quad . \quad . \quad (40)$$

If we take $\delta^2=\frac{\nu^2\rho^2\mathrm{A}_0}{k^2\mathrm{M}_0}$, and solve for $\frac{\mathrm{A}_0}{\mathrm{A}}$, we find

$$\frac{\mathrm{A}_0}{\mathrm{A}}=(1+\delta^2)\,.\,s-\frac{\delta^2}{s}=\frac{\mathrm{M}}{\mathrm{M}_0}-\delta^2\left(\frac{\mathrm{M}_0{}^2-\mathrm{M}^2}{\mathrm{MM}_0}\right) \qquad . \quad . \quad (41)$$

$$\frac{\mathrm{M}_0\mathrm{A}_0}{\mathrm{MA}}=1-\delta^2\left(\frac{\mathrm{M}_0{}^2-\mathrm{M}^2}{\mathrm{M}^2}\right);$$

and since

$$\frac{M_0 A_0(1-\epsilon^2_0)}{MA(1-\epsilon^2)}=1,$$

we have also

$$\frac{1-\epsilon^2}{1-\epsilon_0{}^2}=1-\delta^2\left(\frac{M_0{}^2-M^2}{M^2}\right) \quad . \quad . \quad . \quad (42)$$

These formulæ hold only at a discrete set of points, namely, those points of the orbit which correspond to the perihelion (or aphelion) points of the ρ-orbit; and since the motion in the ρ-orbit comes to a stop there is a last perihelion point, corresponding to the integral part of $\frac{1}{2\pi}\cdot\frac{\alpha}{\kappa}$, which represents the number of complete circuits in the ρ-orbit. Hence in these formulæ τ cannot legitimately be carried to the limit α. The formulæ indicate that the eccentricity increases and has infinity as a limit, while the major axis passes through infinity, becomes negative, and has zero for a limit; and these conclusions are in harmony with the fact that the orbit has an asymptote.

However, δ^2 must be extremely small, and therefore the eccentricity is essentially constant and the major axis is inversely proportional to the mass for any changes in mass which can occur in a star. The electrostatic potential of a positive and negative electron is

$$V=\frac{e^2}{r}.$$

Taking $e=4{\cdot}8\times10^{-10}$, r equal to 2×10^{-13} cm., as given by Millikan for the negative electron, and 6×10^{23} atoms per gram, we have 7×10^{17} ergs per gram; or if we take r equal to 10^{-16}, the radius of the positive electron, we have 14×10^{20} ergs per gram of matter. The theory of relativity gives 9×10^{20} ergs per gram without any models or any uncertainty, and it is no small tribute to the theory of relativity that this perfectly definite figure is entirely reasonable on electrostatic principles. It indicates that the radius, r, at which the electrostatic potential energy is transformed into radiant energy, is slightly larger than Millikan's value of the positive electron. These figures indicate that the sun radiates about $\frac{2}{3}\times10^{-13}$ part of its mass per year, and therefore δ^2 is of the order of 10^{-26}. Hence, if the sun were reduced to one one-thousandth of its present mass, the formulæ

$$\frac{A_0}{A}=\frac{M}{M_0}, \quad \text{and} \quad \epsilon=\epsilon_0$$

would be sensibly correct. Indeed, the term $\delta^2\left(\frac{M_0{}^2-M^2}{MM_0}\right)$ would have the value $\frac{1}{2}$ only after the radius of the sun had been reduced to about a metre.

The problem of two bodies which are growing by gathering in nebulous and fragmentary matter from the space through which it travels was discussed by me in the *American Mathematical Monthly* for October 1919. It is a very different problem from that of declining mass. It, too, leads to an essentially constant eccentricity, but the major axis is inversely

proportional to the cube of the mass, and the period inversely proportional to the fifth power of the mass, so that the short periods of the spectroscopic binary stars are readily accounted for by the hypothesis that a binary or multiple star is the final stage of a planetary system, one or more of the planets having grown in the course of time to starhood. It seems also inevitable that some binary stars will so increase in mass that their components will be drawn within Roche's limit and some kind of a cataclysm will occur. Possibly cepheid variables have such an origin.

Our ideas of cosmology are completely transformed by a recognition of the fact that atoms are evolved in the depths of space, with a resulting absorption of radiant energy ; and that these atoms are again dissolved in the interior of a star, with a consequent liberation of kinetic energy. The property of mass is a temporary property, just as the colour of a flower is a temporary property.

The Effect of Varying Mass on a Binary System.

By J. H. Jeans.

1. In an interesting paper * under the above title Professor E. W. Brown has recently discussed some results I published in *M.N.*, November 1924 (p. 2). He proposes three possible alternative forms of the equations of motion as follows :—

$$\text{mass} \times \text{acceleration} = \text{force} \quad . \quad . \quad (1)$$

$$\text{rate of change of momentum} = \text{force} \quad . \quad . \quad (2)$$

$$d^2/dt^2(\lambda x) = \lambda \times \text{force along the } x\text{-axis} \quad . \quad . \quad (3)$$

where λ is a function of the masses, and says : "There appears to be no information as to which should be chosen if mass is lost in any way other than that given by the classical laws of dynamics."

It may be worth pointing out that the theory of relativity limits us quite definitely to the first alternative, which is the one I used in my paper. Consider a star in space, under no forces, radiating its mass away. To an observer initially at rest relative to the star, the star must obviously remain at rest, for there is no reason why it should start into motion in one direction rather than another. Hence to an observer moving relative to the first observer with a uniform velocity, the star must appear to move with a uniform velocity. Thus when the force is zero, the acceleration must be zero. Of Professor Brown's three hypotheses, the only one which gives zero acceleration when the force is put equal to zero is the first, which I adopted.

From this, as I showed in my paper, it is easy to deduce the relation

$$Ma = \text{constant} \quad . \quad . \quad . \quad . \quad (4)$$

where M is the combined mass and a the semi-major-axis of the orbit.

* *Proc. Nat. Acad. Sci.*, **11**, 274 (1925).

2. I think, however, Professor Brown is in error in deducing the relation a=constant in his equation (2·7). He transforms my original equation quite accurately into

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dr}{dt}\right)^2+\frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)^2-\frac{M_0}{r}\right\}=(M-M_0)\frac{d}{dt}\left(\frac{1}{r}\right),$$

where M_0 is any constant, and puts $-M_0/2a$ for the expression in curled brackets on the left. This is accurate only if $2a$ is taken to mean the major-axis of the orbit which would be described if M were suddenly increased (or reduced) to M_0; not otherwise. With this meaning for a it is not legitimate to replace the average value of $1/r$ by $1/a$, and Professor Brown's proof that a is constant seems to me to fail.

3. The co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of gravity of the system are determined by

$$(m_1+m_2)\bar{x}=m_1x_1+m_2x_2,$$

etc., where m_1, m_2 are the constituent masses and x_1, y_1, z_1 and x_2, y_2, z_2 their co-ordinates at any instant. From the equations of motion Professor Brown derives the relation

$$m_1\ddot{x}_1+m_2\ddot{x}_2=0,$$

which leads to

$$\frac{d^2}{dt^2}(\bar{x})=2\dot{x}\dot{p}+x\ddot{p} \quad . \quad . \quad . \quad . \quad (5)$$

where $x=x_2-x_1$ and $p=(m_1-m_2)/2(m_1+m_2)$.

The simplest case occurs when $\ddot{p}=0$, in which case $\dot{p}$ has a constant value P and equation (5) has the integral

$$\frac{d\bar{x}}{dt}=2xP+V,$$

where V is a constant. Thus the velocity $d\bar{x}/dt$ undergoes small fluctuations (of the order of dp/dt times the linear dimensions of the orbit) about the constant value V. This is in accordance with the theory of relativity which shows, by an argument similar to that already used for a single star, that a binary system acted on by no external forces cannot develop a secular acceleration. This argument shows that even when $\ddot{p}$ does not vanish, $d\bar{x}/dt$ can only vary by minute fluctuations from a steady mean value V.

4. Using equations of motion of the form (1), the areal velocity $r^2\dot{\theta}$ is immediately seen to retain a constant value n throughout the motion. The eccentricity of the orbit being described at any instant is given by

$$1-e^2=\frac{l}{a}=\frac{n^2}{aM},$$

and since aM remains constant (equation (4)), it follows that e remains constant. In my paper I gave this result for the special case in which the equations were integrable, but failed to notice that it is true in the general case. It is important as showing that the high eccentricities of aged binaries cannot be a consequence of age alone; it provides a

direct criterion of the extent to which the orbit has been knocked about by encounters with other systems.

5. In the last paragraph of his paper (§ 9) Professor Brown assigns three main properties to systems governed by equations of the type :

$$\text{mass} \times \text{acceleration} = \text{force},$$

namely, increase of distance, increase of eccentricity, and increase of velocity of centre of mass. It will have been seen that I find myself in disagreement with Professor Brown as regards the last two. He further suggests that the scheme of equations (2) should be chosen in preference to (1), as leading to an increase in the velocity of a single star as well as an increase in the velocity of the centre of mass of a binary system. But either of these increases would violate the principle of relativity, so that they provide reasons against, and not for, the adoption of the second scheme of equations.

A Theory of Stellar Evolution. By J. H. Jeans.

1. In recent papers * I have put forward a series of suggestions and hypotheses which lead to a consistent scheme of stellar evolution. More recently Professor H. N. Russell † has put forward a different train of ideas, and has expounded the scheme of stellar evolution to which they lead. Fundamentally the difference between the two schemes can be traced back to our different conceptions of the nature of stellar matter. My scheme is based on the supposition that it is analogous to a radioactive substance such as uranium in that the transformation of matter into energy proceeds at a rate which is approximately uninfluenced by changes of temperature and density. Russell's scheme, on the other hand, is based on the hypothesis that the rate of transformation depends very largely on the temperature of the matter. For every type of matter he postulates a certain critical temperature—a sort of flash-point —at which " a process comes into play which leads to the actual annihilation of the main mass of the stellar material, with a correspondingly great liberation of energy." If my stellar matter is compared to uranium, Russell's must be compared to gunpowder.

Before it can be accepted, any theory of stellar evolution must pass certain stability tests. These are formulated in §§ 2–8 of the present paper. The result appears to show that stars formed on Russell's model would be violently unstable, and this would seem to rule out his scheme. The stability test further appears to limit the field to schemes of the general character of my own. I have tried to show that this scheme, slightly refined and extended in certain directions, shows promise of explaining the main facts of stellar evolution in so far as they are at present known. A summary of the results obtained, together with

* *M.N.*, Jan., Mar., and June 1925 ; and *Nature*, Feb. 28 and Apr. 4, 1925.
† *Nature*, Aug. 8, 1925.

a general discussion of the problem, is given in §§ 19–22, to which the non-mathematical reader may prefer to turn at once.

The Stability Criterion.

2. Let fig. 1 represent a diagram in which a star's luminosity is plotted against its effective temperature or spectral type, as in Russell's well-known "compass-diagram." Let the line MM′ be supposed to

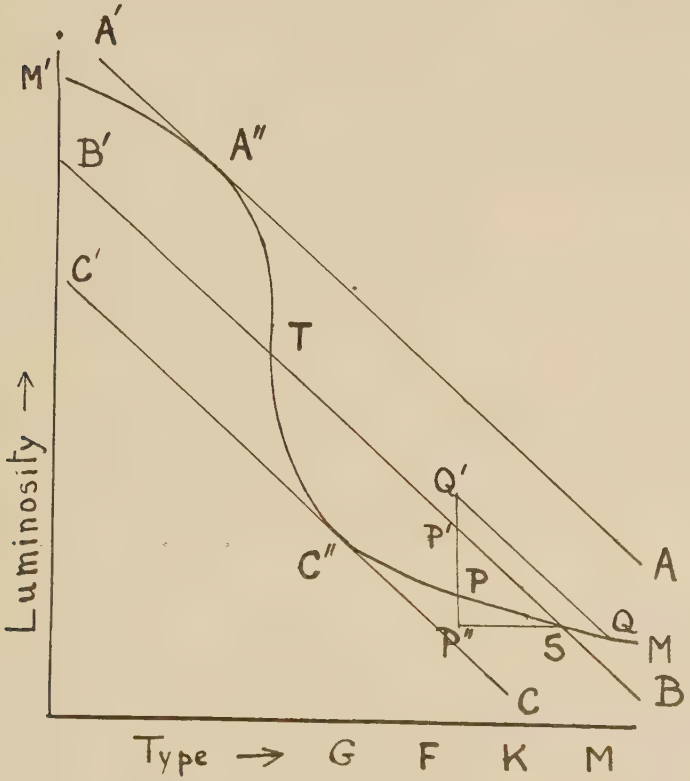

Fig. 1.

represent, in a purely diagrammatic way, the various configurations of equilibrium which are possible for a star of specified mass M.

Let the internal rate of generation of energy at every point of a star be supposed to depend on the temperature T and density ρ at that point, so that a star's total rate of generation of energy will be represented by an integral of the type

$$G = \int 4\pi\rho r^2 f(T, \rho) dr \quad . \quad . \quad . \quad . \quad (1)$$

where $f(T, \rho)$ depends not only on T and ρ but also on the particular type of matter of which the star is made.

We can calculate the value of G for each configuration on the line MM′ and can represent each value of G, measured as a luminosity, by a point P′ either vertically above or vertically below the point P to which it refers. Corresponding to any assumed type of matter, the various points P′ will form a line such as BB′ (which, for diagrammatic simplicity, has been drawn as a straight line). We may call BB′ a " generation line " corresponding to the " emission line " MM′. There is a different " generation line " for each type of stellar material, but the " emission line " is fixed by the mass of the star, except for a small correction arising from differences of distribution of energy-generation in the star's interior. This correction has been studied by Eddington,* but we must be content to disregard it in the present investigation.

The intersections ST of a generation line BB′ and the corresponding emission line MM′ determine the possible configurations of equilibrium for a star of given mass, made of that special type of matter to which the line BB′ refers.

As a star ages and its more readily transformable material becomes exhausted, the line BB′ will move downwards. Corresponding to certain ages of a star the intersections ST may coalesce in two adjacent positions of equilibrium as at A″ or C″. The star can move from one of these configurations to the adjacent one without any forces of restitution coming into play, so that A″ and C″ represent configurations of neutral equilibrium.

3. Let a star, initially in equilibrium, undergo a disturbance such that the element initially at r moves to $r+\delta r$, the values of T and ρ changing to $T+\delta T$, $\rho+\delta\rho$. Since the mass of the shell $4\pi\rho r^2 dr$ remains unaltered, δr and $\delta\rho$ must be connected by

$$\frac{\delta\rho}{\rho}+\frac{2\delta r}{r}+\frac{d}{dr}(\delta r)=0 \quad . \quad . \quad . \quad . \quad (2)$$

If the opacity varies as ρ/T^{3+n}, the original flux of energy H was given by

$$H=-\frac{16\sigma T^{6+n}\mu}{3\kappa\rho^2}\frac{\partial T}{\partial r} \quad . \quad . \quad . \quad . \quad (3)$$

and the disturbed flux is $H+\delta H$, where

$$\frac{\delta H}{H}=\frac{(6+n)\delta T}{T}-\frac{2\delta\rho}{\rho}-\frac{d}{dr}(\delta r)+\frac{d\delta T}{dr}\Big/\frac{\partial T}{\partial r} \quad . \quad . \quad (4)$$

The flux across the sphere originally at r but now at $r+\delta r$ is

$$4\pi(r+\delta r)^2(H+\delta H)=4\pi r^2 H+\delta(4\pi r^2 H),$$

so that the disturbance has resulted in an excess flux dQ into the shell which originally had radii r and $r+dr$, given by

$$dQ=-\frac{d}{dr}\delta(4\pi r^2 H)dr \quad . \quad . \quad . \quad . \quad (5)$$

* *M.N.*, Mar. 1925, p. 408.

Suppose that the disturbance results in an additional generation of energy δG per unit mass per unit time inside this shell. We then have as the equation of energy for the matter inside the shell

$$dQ+4\pi\rho r^2\delta G dr=4\pi\rho r^2 dr C_v\frac{d}{dt}(\delta T)+(p+\tfrac{1}{3}aT^4)\frac{d}{dt}\delta(4\pi r^2 dr),$$

where p is the gas pressure, $\frac{1}{3}aT^4$ the pressure of radiation, and C_v is the specific heat at constant volume of matter and radiant energy together. Since the element $4\pi\rho r^2 dr$ remains of constant mass, $4\pi r^2 dr$ must vary inversely at ρ, so that our equation becomes

$$-\frac{d}{dr}\delta(4\pi r^2 H)dr+4\pi\rho r^2\delta G dr=4\pi r^2 dr\left[\rho C_v\frac{d}{dt}(\delta T)-(p+\tfrac{1}{3}aT^4)\frac{d}{dt}\left(\frac{\delta\rho}{\rho}\right)\right] \qquad (6)$$

The dynamical equation of motion is derived by variation of the standard equation

$$\ddot{r}=-\frac{1}{\rho}\frac{d}{dr}(p+\tfrac{1}{3}aT^4)-\frac{\gamma M_r}{r^2} \quad . \quad . \quad . \qquad (7)$$

and is found, after slight simplification, to be

$$\frac{d^2}{dt^2}(\delta r)=\frac{4\gamma M_r}{r^3}\delta r-\frac{1}{\rho}\frac{d}{dr}[\delta p+\delta(\tfrac{1}{3}aT^4)] \quad . \quad . \qquad (8)$$

Replacing p by its value in terms of ρ and T and using equation (2) to eliminate $\delta\rho$, this becomes

$$\frac{d^2}{dr^2}(\delta r)=\left[\frac{4\pi M_r}{r^3}+\left(\frac{2p}{\rho r}+\frac{1}{\rho}\frac{dp}{dr}\right)\frac{d}{dr}+\frac{p}{\rho}\frac{d^2}{dr^2}\right]\delta r-\frac{1}{\rho}\frac{d}{dr}\left[\left(\frac{p}{T}+\frac{4}{3}aT^3\right)\delta T\right] \qquad (9)$$

Solving symbolically for δT we obtain a relation of the type

$$\delta T=F\left[\delta r,\ \frac{d^2}{dt^2}(\delta r)\right],$$

where F is linear in δr and $\frac{d^2}{dt^2}(\delta r)$, but involves r, ρ, T, and $\frac{d}{dr}$. There is of course a similar relation expressing $d(\delta T)/dt$ as a function of $d(\delta r)/dt$ and $d^3(\delta r)/dt^3$. Substituting these values for δT and $d(\delta T)/dt$ into equation (6) and assuming δG to depend linearly on δT and $\delta\rho$, we obtain a relation of the type

$$\Psi\left[\delta r,\ \frac{d}{dt}(\delta r),\ \frac{d^2}{dt^2}(\delta r),\ \frac{d^3}{dt^3}(\delta r)\right]=0 \quad . \quad . \qquad (10)$$

where Ψ again is linear but involves r, ρ, T, and d/dr.

4. Assuming a time-factor $e^{\lambda t}$, equation (10) becomes

$$\Psi(\delta r,\ \lambda\delta r,\ \lambda^2\delta r,\ \lambda^3\delta r)=0 \quad . \quad . \quad . \qquad (11)$$

which is a differential equation for δr in terms of r. On substituting any solution into the boundary condition which must be satisfied at the surface of the star we obtain an equation in λ

whose roots give the values of λ corresponding to the different free-vibrations of the star. Clearly this equation must be of at least the third degree in λ, and does not retain its form when the sign of λ is changed. It follows that the time-factor $e^{\lambda t}$ can have either of the forms

$$e^{\alpha t}, \quad e^{(\beta \pm i\gamma)t} \qquad . \quad . \quad . \quad . \quad (12)$$

5. Consider the limiting form assumed by a vibration with a time-factor $e^{\lambda t}$ where λ may be of either of the forms just specified, but is assumed to be very large numerically.

The ratio of $\delta T/T$ to $\delta r/r$ will in general depend on λ, and as λ approximates to infinity, this ratio may either remain finite or may become zero or infinite.

If in the limit δT becomes zero in comparison with δr, the last term of equation (9) drops out altogether and the remaining equation may be written as

$$\frac{d^2}{dt^2}(r\delta r)=\left[\frac{4\gamma M_r}{r^3}+\frac{1}{\rho}\frac{dp}{dr}\left\{-\frac{1}{r^3}+\frac{1}{r^2}\frac{d}{dr}\right\}+\frac{p}{\rho}\frac{d^2}{dr^2}\right](r\delta r) \qquad . \quad (13)$$

The equation

$$\frac{d^2}{dt^2}(r\delta r)=\frac{p}{\rho}\frac{d^2}{dr^2}(r\delta r) \qquad . \quad . \quad . \quad . \quad (14)$$

obtained by taking only the first and last terms in equation (13), represents the propagation of spherical waves with a velocity $(p/\rho)^{\frac{1}{2}}$. Equation (13) is the equation of propagation of the same waves modified by the term $4\gamma M_r/r^3$ which represents the tendency to gravitational instability, and by the term in dp/dr which represents the disturbing influence of variations of pressure and density in the star's interior.

Neglecting these variations of pressure and density, equation (14) has solutions of the form

$$\delta r=\frac{A}{r}e^{i(\pm\lambda t\pm\mu r)} \qquad . \quad . \quad . \quad . \quad (15)$$

where $\lambda^2=(p/\rho)\mu^2$, and by combination of such solutions we obtain a solution

$$\delta r=\frac{A}{r}\cos(\lambda t-\epsilon)\sin\mu r \qquad . \quad . \quad . \quad (16)$$

which clearly represents pulsations of the gaseous star. In general, however, the preliminary condition as to the smallness of δT cannot be satisfied.

The case in which δT remains finite in comparison with δr is similar to that just discussed, except that the retention of the final term on the right in equation (9) introduces complications produced by the flow of heat. In place of having the form specified by equation (15), the solution will now have a time-factor of the form $e^{-(\mu\pm i\lambda)t}$, and so will represent pulsations damped by thermal terms. These pulsations would of course be unstable if μ were negative.

Finally, if $\delta T/T$ is infinite in comparison with $\delta r/r$, we find that there can be a solution in which $\delta r/r$ is of the order of $\delta T/\nu^2 T$, equations (6) and (9) reducing to

$$-\frac{d}{dr}\delta(4\pi r^2 H)dr+4\pi\rho r^2\delta G dr=4\pi\rho r^2 dr C_v\frac{d}{dt}(\delta T) \quad . \quad . \quad (17)$$

$$\frac{d^2}{dt^2}(\delta r)=-\frac{1}{\rho}\frac{d}{dr}\left[\left(\frac{p}{T}+\frac{4}{3}aT^3\right)\delta T\right] \quad . \quad . \quad . \quad (18)$$

Since $\delta\rho/\rho$ is of the same order as $\delta r/r$, we may neglect the dependence of δG on ρ, and suppose the whole value of δG to be given by

$$\delta G=B\delta T \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Assuming a time-factor $e^{\nu t}$, equation (17) now becomes

$$-\frac{d}{dr}\delta(4\pi r^2 H)dr=4\pi\rho r^2 dr[\nu C_v-B]\delta T \quad . \quad . \quad (20)$$

The boundary condition requires that δH at the boundary shall be finite, and this is satisfied if $\nu C_v-B=0$ at every point. This requires that B shall be proportional to C_v throughout the star, and there is now a solution with a time-factor $e^{\nu t}$ where $\nu=B/C_v$, for which δr and $\delta\rho$ may be regarded as negligible in comparison with δT. The disturbance here considered is one in which primarily the temperature changes as a consequence of changes in the generation of heat, the flow of heat and mass-motion being merely secondary phenomena. If the disturbances previously considered are described as " pulsatory " disturbances, that just discussed may be described as a " thermal disturbance."

6. We now see that the two types of time-factor which have been seen to be possible in § 5 (formulæ (12)) correspond in a general sort of way to thermal and pulsatory disturbances respectively. In the limiting cases of disturbances of infinite rapidity which we have considered, the correspondence is definite and the line of demarcation between thermal and pulsatory disturbances is perfectly sharp. The definiteness disappears even in the case to which equation (20) refers if C_v is not proportional to B throughout the star, for it can then be shown that wave-motions and mass-displacements must supervene on top of the pure thermal motion. On passing to less rapid disturbances, the line of demarcation must doubtless become still less distinct, and the two possible types of time-factor will only correspond in a very blurred way to the two types of disturbance. The analysis of § 8 has nevertheless suggested the kind of physical interpretation which may be assigned to the two possible types of time-factor.

7. Vibrations with time-factors

$$e^{\alpha t}, \quad e^{(\beta\pm i\gamma)t} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

are stable when α and β are negative and are unstable when α and β are positive. As the evolution of a star proceeds, α, β, and γ must vary continuously, so that changes from stability to instability can occur when α or β vanishes but at no other time.

If α vanishes, equation (6) becomes

$$\frac{d}{dr}\delta(4\pi r^2\mathrm{H})dr=4\pi\rho r^2\delta\mathrm{G}dr \quad . \quad . \quad . \quad . \quad (22)$$

On integrating through the star we obtain

$$\int 4\pi\rho r^2\delta\mathrm{G}dr=\delta(4\pi r^2\mathrm{H}) \text{ at the star's surface} \quad . \quad . \quad (23)$$

which is precisely the condition that defines points such as A″ or C″ in fig. 1. Thus what we may call α-stability can only be lost at points such as A″ or C″.

It is of course possible for the distribution δG to be such that equation (23) is satisfied by the star as a whole without equation (22) being satisfied at every point. We must, however, remember that in drawing our equilibrium line MM′ we disregarded variations of configuration which resulted from different localisations of energy-generation in the star's interior. A point such as A″ represents an infinite number of configurations, all having the same total generation and the same emission, but having different distributions of generation. Thus equation (23) defines the point A″, but for one of the configurations represented at A″ equation (22) will be true. For the present purpose we may regard equations (22) and (23) as being equivalent.

If β vanishes, the time-factors $e^{(\beta\pm i\gamma)t}$ may be replaced by $\cos(\gamma t-\epsilon)$. With this time-factor, equations (2) and (8) require δr, $\delta\rho$, and δT all to be in phase with one another, so that equation (6) can only be satisfied by making the two sides vanish separately, giving rise to the two separate equations

$$4\pi\rho r^2\delta\mathrm{G}=\frac{d}{dr}[\delta(4\pi r^2\mathrm{H})] \quad . \quad . \quad . \quad . \quad (24)$$

$$\rho\mathrm{C}_v\frac{d}{dt}(\delta\mathrm{T})=(p+\tfrac{1}{3}a\mathrm{T}^4)\frac{d}{dt}\left(\frac{\delta\rho}{\rho}\right) \quad . \quad . \quad . \quad (25)$$

For β to vanish, δG must be specified by equation (24). This is formally similar to (22) and gives on integration an equation which is formally similar to equation (23), but it must be noticed that in (24) and (25) δH, etc., refer to changes in a star which is oscillating with a finite frequency $\gamma/2\pi$, whereas equations (22) and (23) referred to motion from one equilibrium configuration to an adjacent configuration of equilibrium.

If equation (24), or its integrated form (23), is satisfied, then equations (8) and (25) specify the particular vibration through which β-stability is lost. They are seen to be precisely identical with the equations for the adiabatic vibration of a sphere of gas.

This explains the way in which β-stability is lost. If a star were generating no energy at all, no pulsation could be strictly adiabatic ; for there would always be some flow of radiant energy, and this would gradually damp the pulsations. Suppose, however, that a sort of Maxwell demon were shut up inside each small element of a star, with a machine for generating energy and instructions to work it in such a way that energy was generated at precisely the rate at which it was

drained away by the flow of energy through the walls of the element. Accurately adiabatic vibrations would now be possible. If the machine, instead of being controlled by an intelligent demon, were left to the play of natural forces, it might either undershoot or overshoot the mark. In the former case the vibrations would decrease in amplitude; in the latter case the amplitude would continually increase and the star would be unstable.

Thus β-stability is lost through a pulsation which, instead of being damped out, begins steadily to increase in amplitude, whereas α-stability is lost by a thermal disturbance which, instead of subsiding, becomes more and more extensive.

8. We proceed to consider the details of loss of β-stability. Equation (25), which is satisfied by the critical vibration through which β-stability is lost, may be replaced by

$$\rho C_v \delta T = (p + \tfrac{1}{3} a T^4) \frac{\delta \rho}{\rho} \quad . \quad . \quad . \quad . \quad (26)$$

in which, as already has been noticed, C_v must include the specific heat of the radiant energy aT^4 per unit volume. If c_v denotes the specific heat of the matter alone,

$$\rho C_v = \rho c_v + 4aT^3.$$

The gas-pressure p is given by $p = c_v(\kappa - 1)\rho T$, where κ is the ratio of the specific heats. On further writing x for T^3/ρ, equation (26) becomes

$$(c_v + 4ax)\frac{dx}{x} = [(3\kappa - 4)c_v - 3ax]\frac{d\rho}{\rho} \quad . \quad . \quad . \quad (27)$$

of which the integral is

$$(3\kappa - 4)\log \rho = \log x - \left(4\kappa - \frac{13}{3}\right)\log [(3\kappa - 4)c_v - 3ax] + a \text{ constant}.$$

For a gas at high temperature with a substantial degree of ionisation, κ approximates to $1\frac{2}{3}$, so that we may suppose $(3\kappa - 4)$ to be positive.

For a star of mass so small that pressure of radiation may be neglected in comparison with gas-pressure, the terms multiplied by a in the above equations may be omitted in comparison with the other terms. We see that x and ρ increase together, the integral reducing to the usual equation for adiabatic motion, $T \propto \rho^{\kappa - 1}$. As the importance of pressure of radiation increases, the terms in a become more important, but x and ρ continue to increase together until a value is reached at which

$$3ax = (3\kappa - 4)c_v \quad . \quad . \quad . \quad . \quad (28)$$

after which an increase in x is accompanied by a decrease in ρ, and conversely.

On restoring its value T^3/ρ for x, it appears that, at the critical stage defined by equation (28),

$$(3\kappa - 4)c_v = 3aT^4 c_v(\kappa - 1)/p,$$

so that

$$\frac{\frac{1}{3}aT^4}{p} = \frac{3\kappa - 4}{9(\kappa - 1)}.$$

With $\kappa = 1\frac{2}{3}$, which is a good enough approximation for our present purpose, the left-hand member becomes equal to $\frac{1}{6}$.

Thus the critical stage defined by equation (28) is reached as soon as radiation-pressure becomes equal to one-sixth of gas-pressure, and this corresponds to a mass of about $2\odot$.

This determination of the critical mass is somewhat rough since we have taken no account of the "specific heat of ionisation," or the work done in changing the degree of ionisation of stellar matter, but we shall assume it to be accurate enough for our present purpose.

We have accordingly seen that for stars of mass less than $2\odot$, ρ and x increase together, while for stars of mass greater than $2\odot$ the reverse is the case. Or, to put it in another way, for $M < 2\odot$, T^3 varies more rapidly than ρ during the progress of an adiabatic vibration, while for $M > 2\odot$, T^3 varies less rapidly than ρ.

One important deduction from this will be needed very shortly. In a

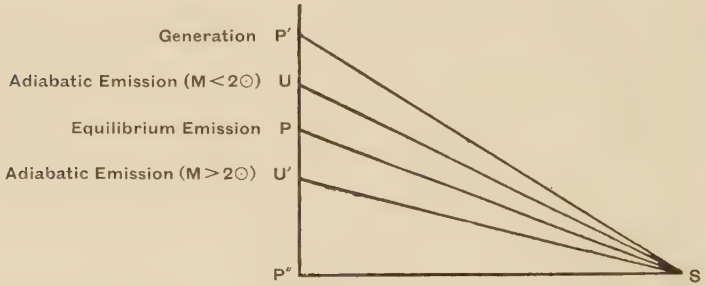

FIG. 2.

homologous expansion or contraction, $4\pi r^2 H$, which becomes equal to the emission of radiation E as the surface is approached, varies as $T^{n+7}\rho^{7/3}$ (*cf.* equation (3)). Thus for a star of mass less than $2\odot$, E varies more rapidly than T^n, whereas for a star of mass greater than $2\odot$, E varies less rapidly than T^n. The function T^n gives the dependence of E on T as we pass along a series of equilibrium configurations.

9. We are now in a position to formulate the general stability criterion.

Let fig. 2 represent a magnification of that part of fig. 1 which is in the immediate neighbourhood of the equilibrium configuration S. Let SP be the line of equilibrium configurations (emission line) through S, and let SP′ be the corresponding generation line.

If the star's temperature changes from that of S to that of P, we suppose the star's generation of energy to change by an amount represented by P″P′. At configurations such as A″ and C″ in fig. 1 the lines SP and SP′ coincide, and we have seen that there is a change of α-stability at such configurations.

As a star's evolution progresses, the slope of the lines SP and SP′ continually changes, and a change of α-stability occurs whenever the lines cross over one another. If a star starts life in a stable configuration,

it remains stable, as regards α-stability, until the line SP, SP′ cross one another, after which it becomes unstable. The consideration of any single case will tell us which direction of crossing represents the change from stability to instability. The case discussed in § 5 showed that extreme slope of the line SP′ upwards to the left corresponded to extreme instability. Thus the general law must be that α-stability is lost as soon as the slope of the generation line SP′ exceeds that of the emission line SP, slope being measured so that the slopes of SP and SP′ in fig. 2 are reckoned positive.

For an adiabatic oscillation of the type considered in § 8, let the star's temperature again change through a range represented by SP″, and let SU be the corresponding emission line. Then we have seen (§ 7) that β-stability is lost when the lines SU and SP′ cross one another.

For a star of mass less than $2\odot$ we have seen (§ 8) that the emission varies more rapidly with the temperature along SU than along SP, so that U is necessarily above P. Thus a change of β-stability cannot occur until P′ is above P, and by this time the star is already unstable through loss of α-stability.

Thus a star of mass less than $2\odot$ necessarily first loses its stability through loss of α-stability, *i.e.* through a thermal disturbance and not through a dynamical pulsation. The criterion for absolute stability is simply that the star shall not have lost its α-stability, and this merely requires that the generation line SP′ shall have less slope than the emission line SP. Analytically, if E is the emission of energy and G the corresponding generation of energy along a series of equilibrium configurations for a star of given mass M, the star first loses its stability when

$$\frac{d(\mathrm{E}-\mathrm{G})}{d\mathrm{T}} \text{ becomes negative} \quad . \quad . \quad . \quad (29)$$

For a star of mass greater than $2\odot$, let SU′ be the line of emission for adiabatic pulsations. We have seen that when $\mathrm{M}>2\odot$, SU′ is necessarily below SP. Such a star must first become unstable through loss of β-stability, *i.e.* through a pulsation and not through a thermal explosion. The criterion for absolute stability is now that the slope of the generation line SP′ shall be less than that of the adiabatic emission line SU, a condition which is more stringent than that for a star of mass less than $2\odot$. Analytically a star of mass greater then 2M first loses its stability when

$$\frac{d(\theta\mathrm{E}-\mathrm{G})}{d\mathrm{T}} \text{ becomes negative} \quad . \quad . \quad . \quad (30)$$

where θ is a numerical multiplier depending on the star's mass, which may be positive or negative, but is always algebraically less than $+1$ and becomes equal to $+1$ when $\mathrm{M}=2\odot$.

10. As an instance of the use of these criteria, let us notice that in fig. 1, as drawn, the only possible range of stable configurations is A″TC″; all others are necessarily unstable. If the star's mass is less than $2\odot$, the actual range of stability will be A″TC″, but if the mass is greater than $2\odot$, the range will be less than this.

Consider a star of mass less than $2\odot$ and suppose that, owing to the rapid exhaustion of easily transformable material, the line BB′ is falling at a greater rate than the line MM′. In such a case the star may evolve through a series of stable configurations such as A″TC″ and may then fail to find a further configuration of equilibrium through the two lines having parted company at C″. Static equilibrium is no longer possible, and dynamical motion ensues. From the point of view of the equilibrium of a single star of given mass, a configuration such as C″ is merely one of neutral equilibrium, but from the evolutionary point of view such a configuration, precisely as in the corresponding dynamical theory of Poincaré, marks the beginning of a cataclysm.

11. We have so far supposed the rate of generation of energy in any element of a star to be determined entirely by the temperature and density of the element at the moment. Suppose, however, that changing T and ρ to T′ and ρ' does not immediately effect a change but merely starts a change, such that δG gradually changes from the value $f(T, \rho)$ appropriate to the former state to the rate $f(T', \rho')$ which is appropriate to the latter state. Such would, for instance, be the case if the transformation of matter into energy took place through the formation of short-lived intermediate products.

For the sake of precision, let us assume the rate of change to be proportional to the discrepancy between the momentary and final rates of generation of energy. Then at a time t after the change from T, ρ to T′, ρ' occurred, the rate of generation of energy must be of the form

$$e^{-\kappa t}f(T, \rho)+(1-e^{-\kappa t})f(T', \rho') \quad . \quad . \quad . \quad (31)$$

while the rate of generation of energy in the whole star will be

$$G=e^{-\kappa t}\int 4\pi\rho r^2 f(T, \rho)dr+(1-e^{-\kappa t})\int 4\pi\rho r^2 f(T', \rho')dr \quad . \quad (32)$$

The special cases of κ very large and κ very small admit of easy discussion.

If $1/\kappa$ is small in comparison with the star's periods of vibration, it is easily seen that the discussion already given remains valid.

If $1/\kappa$ is large in comparison with the star's periods of vibration, the discussion of ordinary stability will be the same as it would be if the generation lines were horizontal. If a star in a configuration such as S in fig. 1 suddenly increases its rate of generation of energy by P″P, the star will at first settle down into a position of equilibrium at, or of oscillations about, the neighbouring configuration P whose emission of energy is appropriate to the star's enhanced rate of generation. At first the ordinate of P will represent the star's total generation of energy, but the rise of temperature involved in moving from S to P will start a gradual further generation of energy, which, if the star were held at rest, would in time reach a limit P″P′. As this additional generation of energy gradually makes itself felt, the star will behave as it did under the original increase of generation P″P, and so will move still further away from S. Now the final generation line in our diagram is QQ′, where Q′P′=PP″, so that the only configuration of equilibrium which is permanently possible for the star is Q. Thus the star, which is steadily

moving away from Q, can never attain permanent equilibrium and must be regarded as unstable. Had the slope of the final generation line SP′ been less than that of the emission line SP, the permanent equilibrium configuration Q would have been on the same side of S as P, and the equilibrium would have been stable.

The foregoing argument only covers the problem of α-stability, but can immediately be modified so as to apply to β-stability. In each case we see that the criterion is the same as before, provided the generation line is drawn so as to represent the final rate of generation of energy. In the analytical formulæ (29) and (30) also, G must be taken to mean the final rate of generation of energy, *i.e.* the limit given by formula (32) on putting $t = \infty$.

12. Three applications of the stability criterion are of interest.

(1) If Eddington's mass-luminosity law were accurately true, the emission lines would be straight horizontal lines, so that a star could only be stable if an increase in its temperature resulted in a decrease in its generation of energy.

(2) According to Russell's scheme, the emission lines are approximately horizontal, while the generation lines run violently uphill in the neighbourhood of a certain critical temperature (generally about 30,000,000° C.). Russell considers that when a star approaches this critical temperature the increased generation of energy results in a heating of the star which causes an expansion which in turn causes a cooling which checks any further generation of energy. Our analysis has not borne out this view ; the general stability-criterion has shown that all stars whose central temperatures approximated to this temperature would be violently unstable. The analysis of § 5 indicates the sequence of events which would occur when the centre of a star of this type first reached the critical temperature ; we see that it would be very similar to that occurring when the centre of a barrel of gunpowder reached the flash-point of the powder.

(3) According to the rough preliminary theory I have sketched out elsewhere,* the rate of generation of energy is supposed to be approximately independent of the density and temperature of the matter, so that the generation lines are approximately horizontal. All configurations will now have α-stability if the emission lines slope upwards to the left, as is normally the case, although in the case of stars of mass greater than $2\odot$, the criterion for β-stability may be somewhat more stringent, and the emission lines may need to have a considerable slope upwards to secure stability.

Emission Lines.

13. Let us now examine what arrangement of emission lines will represent actual stellar conditions. To a first approximation the emission lines may be regarded as straight lines sloping upwards to the left, as shown in my diagram in *M.N.*, Mar. 1925 (p. 401). A better approximation is obtained by taking into account the variations of ionisation, and consequently of effective molecular weight, in the star's

* *Nature*, Apr. 4, 1925, and *M.N.*, Mar. 1925.

interior. When this is done, my investigation * gives as the equations of the emission lines

$$4s \log T + 0{\cdot}4(s+\tfrac{1}{2})m + (2s+4) \log \mu = \text{constant} \quad . \quad . \quad (33)$$

A correction corresponding to the term in log μ had previously been given by Eddington. Russell, making use of Eddington's correction, finds the shapes of the lines in the region in which they slope upwards to the left to be that shown in fig. 3,† but our equation (33) shows that the slope of the curves depends largely on s and so cannot be determined with very great accuracy.

14. Since the slope of these curves is nowhere great, stability would be impossible if the slope of the generation curves were great. This encourages the hypothesis that the rate of generation of energy is, to a first approximation, unaffected by temperature and density. Making this hypothesis, the generation curves become horizontal lines, and the regions in which the emission lines slope upwards to the right represent unstable configurations. The boundary of this region is denoted by the thick curve on the right in fig. 3 (p. 930).

The position of this curve shows that M-type stars of magnitudes between about +4 and −2 would be unstable. There is a similar but smaller region of instability for spectral types to about K4, so that the region of instability has at least some resemblance to our general idea of the gulf between the giants and dwarfs. It is suggested that the reason why this gulf is observed is that stars could not exist stably in this region. The investigations of Adams and Joy, Strömberg and Burwell ‡ suggest that our boundary curve ought to be pushed somewhat downward, but we have not attempted to determine its position with great accuracy.

The curve just discussed marks the limit of α-stability, but stars of mass greater than about $2\odot$ may have lost their β-stability even when they are to the left of this curve. We ought in fact to draw another curve to represent the limit of β-stability; this would leave the former curve at about $M=2\odot$ and would cut off a V-shaped area enclosing giants of the latest spectral types. Stars in this region would be possessed of α-stability but not of β-stability. They would, according to our analysis, be liable to develop pulsations which might increase indefinitely in amplitude. On the other hand, it seems likely that in nature viscous or similar forces might come into play to check the unlimited increase of these pulsations, and possibly β-instability may amount to nothing more than a tendency to develop pulsations of large but finite amplitude. If so, the almost universal variability of the late-type giants may be merely a result of their β-instability.

15. The general trend of the path which would be followed in fig. 3 by a star as its evolution proceeds has been discussed in *M.N.*, March 1925 (p. 401). From the shape of the evolutionary curves there shown, it is clear that stars moving in the stable part of the diagram may

* *M.N.*, Mar. 1925, p. 397, equation (81).

† I have copied these curves from Russell's diagram in *Nature* (Aug. 8, 1925), leaving out those parts where the slope is downwards to the left.

‡ *Ap. Journ.*, **46** (1917), p. 313, and **53** (1921), p. 13.

encounter the curve of limiting stability which bounds the diagram on the right. When this occurs cataclysmic motion must ensue.

The general nature of this cataclysmic motion can easily be seen. Immediately after crossing the curve of limiting stability, a star is radiating more energy than it generates, so that it draws on its gravitational potential energy and shrinks rapidly. The general theory of equilibrium configurations indicates that the star will be unable to find any new position of stable equilibrium so long as the ordinary gas laws hold. We must accordingly suppose the star to shrink until Boyle's law begins to fail, after which it may, in accordance with Eddington's theory of the white dwarfs, find a configuration of stable equilibrium as a white dwarf. Thus we may say that the stars which ought to fill the gulf between giants and dwarfs are absent because they have shrunk into white dwarfs.

16. A reservation must, however, be made. A normal star, in shrinking into a white dwarf, increases its density enormously, and so also, on account of conservation of angular momentum, its velocity of rotation. Unless the star had exceedingly little rotation at the outset, it will hardly escape fission before reaching the white-dwarf state. An argument given elsewhere * shows that fission is accompanied by the advance of each constituent to a much earlier spectral type. Such an advance would be represented in our diagram (fig. 3) by a move to the left, suggesting that before the white-dwarf state is actually reached, fission may throw one or both of the components back into the stable region. We return to this possibility later.

Generation Lines.

17. Let us next consider the shape of the generation lines. According to the preliminary scheme I have already sketched out (*Nature*, December 26, 1924, and February 28, 1925), the generation of energy inside a star is caused by the falling of electrons into orbits of zero quantum-number. Although other assumptions could be made, it seems natural, in a preliminary treatment, to assume that the generation of energy in matter of any given type is proportional to the number of nuclei which have electrons describing quantum-orbits about them, *i.e.* to the number of atoms which are not fully ionised. Some such assumption at least seems necessary to explain the facts.

On this view the generation of energy in a star must fall off when its centre reaches those temperatures at which its atoms begin to be entirely stripped of their electrons. To take a simple artificial case, let us suppose that a star's generation of energy remained unaltered until half its nuclei had been stripped bare of electrons and then fell quite sharply to zero. Clearly the stars in our diagram would then be bounded on the left by the curve representing configurations in which exactly half of the nuclei are stripped bare. The introduction of our hypothesis has moved all the stars which would otherwise have been on the left of the 50 per cent. curve and has crowded them on to the curve. It is suggested that a generalisation of this conception explains the " main-sequence " of Eddington and Hertzsprung. We proceed to exact analysis.

* *M.N.*, June 1925, p. 797.

18. The degree of complete ionisation which corresponds to given values of T and ρ is given by a formula of R. H. Fowler : *

$$\log\left(\frac{x}{1-x}P_e\right)=-\frac{\chi_1}{RT}+\tfrac{5}{2}\log T+\log\frac{(2\pi m)^{\frac{3}{2}}R^{\frac{5}{2}}\sigma}{h^3}-\log b(T) \quad . \quad (34)$$

where x is the fraction of atoms completely ionised, P_e is the partial pressure corresponding to all the free electrons present, χ_1 is the last energy level, σ is the number of valency electrons in the atom in corresponding orbits, and $b(T)$ is the "partition-function." In the present problem we may put $b(T)=1$, $\sigma=2$, $P_e=R\rho T/2H$ where H is the mass of the hydrogen atom, and $\chi_1=hn^2N$ where N is Rydberg's number, this being the negative energy of a one-quantum orbit in an atom of atomic number n. Equation (34) now assumes the form

$$\log\left(\frac{x}{1-x}\rho\right)=-\frac{hn^2N}{RT}+\tfrac{3}{2}\log T+\log\frac{4H(2\pi mR)^{\frac{3}{2}}}{h^3}.$$

Transforming the logarithms from base e to base 10, and inserting numerical values for h, N, R, H, and m, we obtain

$$\log_{10}\left(\frac{x}{1-x}\rho\right)=-\frac{6{\cdot}83\times10^4}{T}n^2+\tfrac{3}{2}\log_{10}T-7{\cdot}790 \quad . \quad (35)$$

From this we obtain the following numerical values for the quantity on the left-hand side :—

Values of $\log_{10}\left(\frac{x}{1-x}\rho\right)$.

T= / n=	10^6.	$10^{6\cdot5}$.	10^7.	$10^{7\cdot5}$.	10^8.	$10^{8\cdot5}$.
10	− 5·62	− 0·20	2·03	3·24	4·14	4·95
20	−26·11	− 6·68	−0·02	2·58	4·01	4·93
30	−60·26	−15·48	−3·43	1·48	3·79	4·89

To a sufficient degree of accuracy for our present purpose we may imagine all the stars arranged according to Emden's solution, $k=\frac{4}{3}$, which corresponds to Eddington's earliest approximations. Such a star of radius 6·90 can be divided into five layers of equal masses $\frac{1}{5}$M by spheres of radii 1·22, 1·72, 2·20, and 2·90. At these dividing spheres the temperatures are 0·80, 0·66, 0·53, and 0·38 times the central temperature, and the densities are 0·51, 0·29, 0·15, and 0·055 times the central density. If $\bar{\rho}$ is the mean density of the star by volume, the central density is $54{\cdot}4\bar{\rho}$. The mean temperatures and densities by mass in the five layers are now found to be as follows :—

Range of r.	Mean T/T_c.	Mean $\rho/\bar{\rho}$.	$\log\left(\frac{x}{1-x}\rho\right)$.	$x(\bar{\rho}=2{\cdot}1)$.
0 to 1·22	0·88	43	2·38	0·73
1·22 „ 1·72	0·74	22	2·07	0·72
1·72 „ 2·20	0·59	12	1·63	0·63
2·20 „ 2·90	0·44	5·4	0·96	0·44
2·90 „ 6·90	0·19	1·6	−2·67	0·00
Mean	0·57	17	..	0·50

* *M.N.*, **83**, 407.

The values of $\log\left(\frac{x}{1-x}\rho\right)$ are calculated for the values $n=20$ and $T_c=10^{7\cdot5}$; those in the last column assume the further value $\bar{\rho}=2\cdot1$. The average value of x for the five shells is seen to be precisely one-half, so that a star with a central temperature of 31,600,000° and a mean density of 2·1 (with $n=20$) would have half of its atoms stripped bare of electrons. Such a star comes just about on the "main-sequence."

Repeating the calculation with $T_c=10^{7\cdot25}$ we find that half ionisation corresponds to a density of about 0·1; with $T_c=10^8$ the requisite density is of the order of 100, and with $T_c=10^{8\cdot5}$ of the order of 1000.

Similar calculations could be made for other values of x, but results of sufficient accuracy for our present purpose can be obtained by the differentiation of equation (35). The curves representing 50, 90, and 99 per cent. complete ionisation are found to lie approximately as shown in fig. 3.

Summary and Discussion.

19. Fig. 3 represents a diagram in which a star's absolute magnitude is plotted against its spectral type. Reasons have been adduced for thinking that stars are necessarily excluded, on physical grounds, from the regions, both on the right and on the left in this diagram, which have been left blank. The remainder of the diagram may be roughly compared in shape to the American continent, giant stars inhabiting North America (with the super-giants in Canada) and dwarf stars inhabiting South America.

It is suggested that the region which corresponds to the Atlantic Ocean is devoid of stars because all the configurations there are unstable. General considerations have shown that an increase in a star's temperature cannot result in any greatly increased generation of energy, for if it did all stars would be unstable. If for simplicity the rate of generation of energy is supposed to be entirely independent of the condition of the star's interior, then the limit of stable configurations is marked by the Atlantic coast in our diagram. It must, however, be remarked that it is not possible to determine the position of this curve with great precision.

Stars which are located on the northernmost parts of the Atlantic coast have been found to suffer from a tendency to develop pulsations which, according to our rough analysis, ought to increase indefinitely in amplitude. Possibly a more refined analysis would show that other forces come into play and limit these pulsations to finite, although still large, amplitudes. If this can be shown to be the case we have a ready explanation of the variability which appears to affect practically all red giants: otherwise we must suppose that stability considerations rule out not only the Atlantic Ocean but also the northern parts of the Atlantic coast as well.

Calculation shows that stars lying on the Pacific coast or in the Pacific Ocean in our diagram would be almost completely resolved, on account of their high internal temperature, into bare nuclei and free electrons. On the assumption that the atoms are all of atomic number

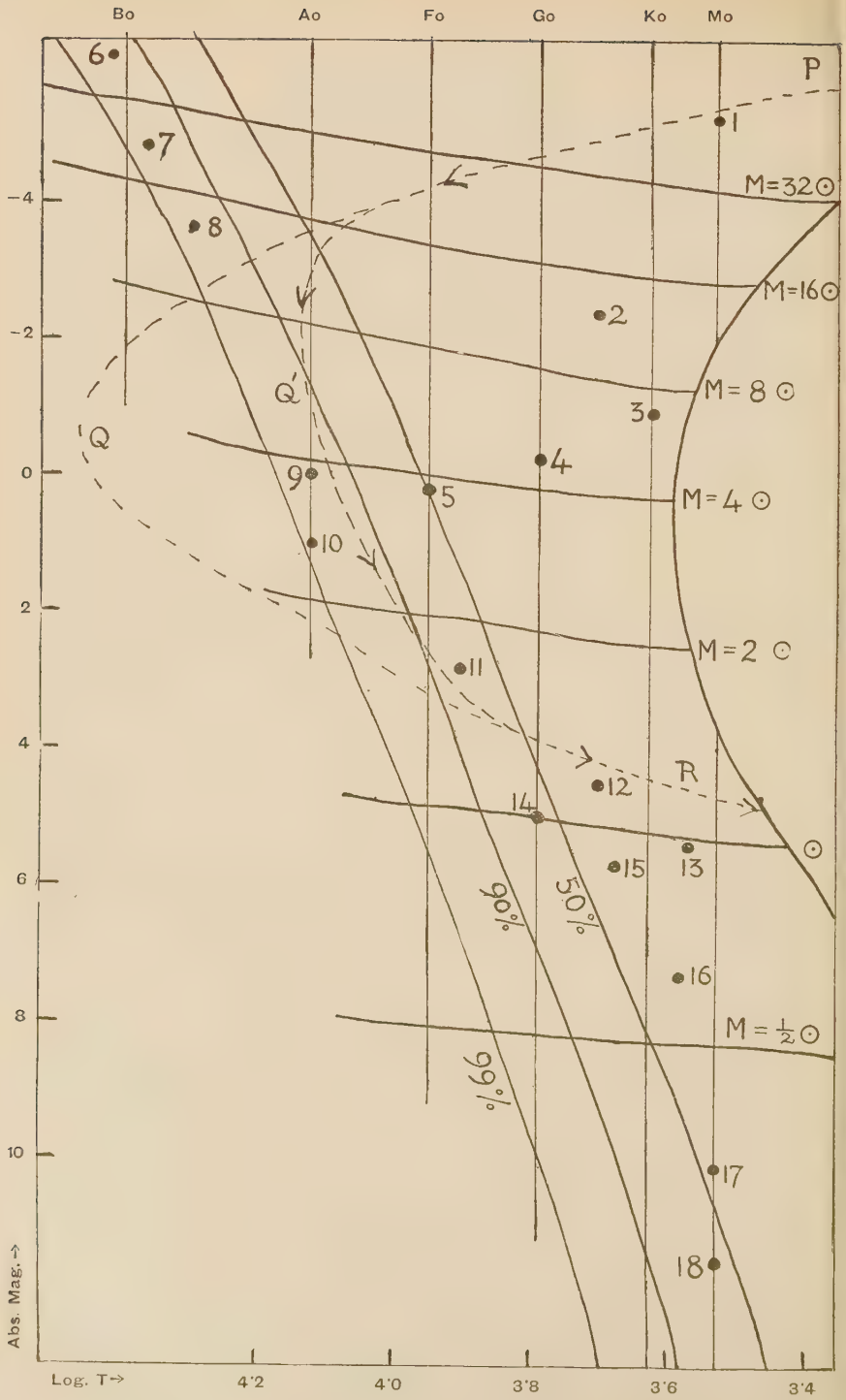

FIG. 3.—The stars, which are those shown by Russell in his diagram, are as follows: 1, Antares. 2, δ Cephei. 3, Arcturus. 4, 5, Capella. 6, Plaskett's star. 7, V Puppis. 8, γ Cygni. 9, β Aurigæ. 10, Sirius. 11, Procyon. 12, 13, α Centauri. 14, Sun. 15, 16, ξ Bootis. 17, 18, Kruger 60.

20, the three parallel lines down the Pacific coast represent configurations of 99, 90, and 50 per cent. total ionisation respectively. It accordingly appears to be an actual fact that stars do not exist in which the total ionisation exceeds a certain limit, let us say 99 per cent., and it is suggested that this is because excessive ionisation inhibits the generation of energy. If, as has previously been suggested, energy is generated by the dropping of an electron from a one-quantum to a zero-quantum orbit, then bare nuclei should have no capacity for generating energy. A star with 75 per cent. of its nuclei stripped bare is, on this hypothesis, generating only a quarter of the maximum energy of which it is capable; it is, so to speak, working with only a quarter of its cylinders in action.

On this hypothesis a star can never cross the Pacific coast in our diagram. With each step west its rate of generation of energy decreases, so that before 100 per cent. of total ionisation is reached, a state of equilibrium is found in which the star's diminished rate of generation of energy is equal to the rate of radiation demanded by its mass and spectral type.

20. Eddington has recently * calculated the diminution in the radiation to be expected from a star of given mass and given spectral type when the generation of energy is confined to its central regions. He finds that such a star may be up to 0·97 m. fainter than if the generation of energy were distributed evenly throughout its mass. Our present hypothesis leads to conditions which are the reverse of those considered by Eddington; it requires the radiation of a highly ionised star to come largely from its outer regions, so that its radiation ought to be greater than would be calculated from its mass and spectral type on the supposition of energy being generated uniformly throughout the star's mass. This consideration would require the equilibrium lines for stars of constant mass to turn up north somewhat sharply on approaching the Pacific coast. Such turns have not been exhibited in fig. 3: the curves there drawn are uncorrected for concentration of energy-generation near the surface.

In Eddington's tables of stars there are a number of pairs of stars of approximately equal mass. In every reliable case the more highly ionised star is substantially brighter than the less highly ionised, the former generally being brighter than it ought to be on the supposition of uniformly distributed generation of energy. We have, for instance:

$$\begin{cases} \text{Procyon F5}(1{\cdot}13\odot) & m=2{\cdot}92 \\ \zeta\ \text{Herculis } b\text{G0}(1{\cdot}09\odot) & m=3{\cdot}20 \\ \alpha\ \text{Centauri } b\text{G5}(1{\cdot}13\odot) & m=4{\cdot}53 \end{cases}$$

$$\begin{cases} \text{Z Vulpec } b\text{B3}(5{\cdot}24\odot) & m=-2{\cdot}70 \\ \text{V Oph. } b\text{B9}(5{\cdot}36\odot) & m=-0{\cdot}47. \end{cases}$$

Possibly the observational material is too unreliable for a single instance to mean much, but it is perhaps not without significance that every instance seems to point in the same direction. The lines of equilibrium configurations for stars of given mass do in actual fact appear to turn up to the north as the "Pacific coast" is approached,

* *M.N.* **85**, 408.

and the most obvious interpretation of this is that the generation of energy is concentrated near the star's surface, so that the generation of energy must be inhibited in the central regions of a star as they become very hot.

21. Making the preliminary supposition that the generation of energy is independent of physical conditions of density and temperature, I have shown * that the evolutionary path of a typical star in our diagram (fig. 3) would be an approximately parabolic curve having its axis nearly horizontal and its vertex to the left. If such a parabolic curve is found to go so far west in our diagram as the Pacific slope, the hypothesis on which it has been calculated needs revision, and, as a result of this revision, the nose of the parabola must be blunted until it lies entirely within the land-area of our diagram. The unadjusted curve PQR must be replaced by PQ'R.

It now appears that a typical star will be born as a super-giant, probably with a surface-temperature too low for visibility, and possibly as a variable undergoing violent pulsations. It will gradually move westward in our diagram until it reaches the region of high ionisation which we have called the Pacific slope ; when it reaches this region, it turns to the south-east and travels down the coast. At this period the centre of the star is a pool of fully ionised matter which is entirely inert as regards generation of energy.† The extent of this pool will in time reach a maximum, after which it declines and finally becomes insignificant. When this stage is reached, the star leaves the Pacific coast and travels across the continent in a south-easterly direction, ultimately falling into the Atlantic. As soon as it touches the Atlantic coast-line instability sets in and dynamical motion ensues. The star rapidly contracts, and only regains stability when its density is so high that the ordinary gas laws no longer hold : it is now a white dwarf.

22. Such would be the whole evolutionary course for a non-rotating star. In this course its density has increased from perhaps 10^{-7} to 10^{+5}; in any case it must have had an extraordinary absence of initial rotation to have reached white-dwarf density without fission.

Neglecting the correction required by the possibility of complete ionisation, I have shown ‡ that fission would, in general, increase a star's surface-temperature enormously. Calculation showed that the surface-temperature might be increased from three- to ten-fold, so that the representative points of the two constituents formed by fission are likely to be thrown right out into the Pacific Ocean. When ionisation is taken into account we must rather suppose that they are thrown across on to the Pacific coast, where they start life with a large central pool of inert (fully ionised) matter. In accordance with this we find that the extreme edge of the Pacific coast is inhabited by newly formed (eclipsing or early spectroscopic) binaries.

* *M.N.*, Mar. 1925, p. 401.

† As an interesting consequence, which does not directly concern us here, the central pool of such a star would be at a uniform temperature. Probably a large part of the interiors of all stars on the main sequence approximate to isothermal equilibrium.

‡ *M.N.*, June 1925, p. 797.

As this central pool disappears, each constituent strikes out eastward across the continent ; in our diagram α Centauri (12 and 13) and Kruger 60 (17 and 18) provide instances of stars at this stage of their careers. If a star has already broken up once by fission, considerations of density make it improbable that a second fission can occur until the onset of the rapid increase of density which accompanies the fall into the Atlantic Ocean. Here fission is more likely than not, and its effect may be to put either one or both constituents back on to the land-area. The process may then repeat itself again, but in any case the ultimate result of the evolutionary process must be a family of white dwarfs gradually fading into darkness and melting into extinction.

On a Theorem of v. Zeipel on Radiative Equilibrium.

By J. H. Jeans.

1. I am greatly indebted to Professor v. Zeipel for the amplification and fuller explanation he has given * of the proof of his theorem, but have to confess that it seems to me as improbable as ever that the theorem has any astronomical significance.

2. Professor v. Zeipel starts with Schwarzschild's equation

$$\frac{dI}{d\sigma}+I=\mu T^4 \qquad . \quad . \quad . \quad . \quad (1)$$

which is exact, σ denoting the optical mass from the surface of the star to a point in the direction to which I refers, so that

$$\frac{d}{d\sigma}=\frac{1}{k\rho ds}. \qquad . \quad . \quad . \quad . \quad (2)$$

He derives an exact solution in the form

$$I=\mu T^4-\frac{d}{d\sigma}(\mu T^4)+\frac{d^2}{d\sigma^2}(\mu T^4)-e^{-\sigma}\int_0^\sigma \frac{d^3}{d\sigma'^3}[\mu T^4(\sigma')]d\sigma'$$
$$+e^{-\sigma}\left[I_0-\mu T^4+\frac{d}{d\sigma}(\mu T^4)-\frac{d^2}{d\sigma^2}(\mu T^4)\right]_{\text{at surface}} \qquad (3)$$

In the interior layers of the star σ is so large that $e^{-\sigma}$, although everywhere finite, is extremely small. Professor v. Zeipel accordingly omits the last two terms on the right of equation (3), thereby reducing the solution to the first three terms, a form first given by Milne.

This procedure is clearly legitimate only if the terms multiplied by $e^{-\sigma}$ are zero or finite. From (2) each differentiation with respect to σ introduces a factor $1/\rho$ (or more accurately $1/\rho^2$, since k itself varies as ρ), so that we should *à priori* expect the term in square brackets in the last line of equation (3) to become infinite at the surface to the order of

* *M.N.*, June 1925, p. 678.

$1/\rho^2$ (or $1/\rho^4$). Thus Professor v. Zeipel's assumption that the remainder term in (3) may be neglected is not *à priori* justifiable, and I cannot see that he makes any attempt to justify it.

3. In §§ 2–7 of my original paper * I gave analysis which was intended to investigate the question from a different angle in such a way that the particular difficulty associated with the remainder term in (3) did not arise. The results indicated that the error in Professor v. Zeipel's equation for ϵ was of the order of the principal terms, so that his results were astronomically of no significance. I do not think Professor v. Zeipel is justified in dismissing this analysis summarily, as he does, by the statement that his theorem only applies to stars in which the level surfaces are neither plane nor spherical. At this stage of the discussion we were not considering his theorem, but were concerned only with the differential equation connecting ϵ with the way in which μT^4 varied from point to point inside a star, and this must be the same for all shapes of stars.

4. Apart from this differential equation and its difficulties, there appear to be other reasons why the theorem does not apply to an actual star.

The equation of mechanical equilibrium in the star's interior is

$$d(p+\tfrac{1}{3}aT^4)=\rho d\Phi \quad . \quad . \quad . \quad . \quad (4)$$

where $\Phi=\Omega+\frac{1}{2}(x^2+y^2)\omega^2$, Ω being the gravitational potential and ω the angular velocity of the star's rotation. The level surfaces $\Phi=$cons. must coincide with the surfaces of constant density $\rho=$cons., and also with the surfaces of constant total pressure

$$p+\tfrac{1}{3}aT^4=\text{cons.} \quad . \quad . \quad . \quad . \quad (5)$$

In general, the gas-pressure depends on T, ρ, and on certain constants of the matter, say m_1, m_2, . . ., these denoting the mean molecular weight, etc., of the gas, say

$$p=f(\rho, T, m_1, m_2 \ldots),$$

so that along the level surfaces $\rho=$cons. we have also

$$f(\rho, T, m_1, m_2 \ldots)+\tfrac{1}{3}aT^4=\text{cons.} \quad . \quad . \quad (6)$$

Professor v. Zeipel assumes the "fundamental hypothesis that the nature of the gas is constant over every level surface," † so that m_1, m_2 are constants over every level surface, from which of course it follows that T also is constant over every level surface. If this were so the temperature would have a constant value T_0 over the outer surface of the star (say $\Phi=\Phi_0$) and another constant value T_1 over an adjacent equipotential (say $\Phi=\Phi_1$) at a short distance inside the star. The flux of radiation at any point of the star's surface would, to a good approximation, be proportional to

$$\frac{T_1-T_0}{\delta n} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

* *M.N.*, April 1925, p. 526.

† *M.N.*, Suppl., 1924, p. 666

where δn is the normal distance between the two surfaces at that point. This flux of energy must, however, have the constant value $\frac{1}{4}a\mathrm{C}\mathrm{T}_0^4$ at every point of the star's surface. Thus δn must be constant over the star's surface which, since δn is the distance between two adjacent level surfaces, means that gravity must be constant over the star's surface.

In general, gravity will not be constant over the surface of a rotating star, and if the star is rotating rapidly, the variations in gravity will easily outbalance those which could be attributed to imperfections in our approximation (7). Thus for a rapidly rotating star Professor v. Zeipel's "fundamental hypothesis" will not be true, and there is, so far as I can see, no reason for expecting it to be true for a slowly rotating star.

I do not think Professor v. Zeipel's treatment of the equations of radiation can be justified, but even if it could, it seems to me that his theorem would lose all interest through his "fundamental assumption" not being satisfied in an actual star. The surfaces of constant temperature will coincide with those of constant density so long as the star is at rest, but in general the two families of surfaces will become distinct as soon as the star begins to rotate. As soon as the theorem becomes applicable, the fundamental assumption which underlies it fails.

Note on the Relations between the Mass, Temperature, and Luminosity of a Gaseous Star. By H. N. Russell.

Jeans * and Eddington † have recently derived certain formulæ connecting the luminosity of a gaseous star with its mass and temperature, which these investigators appear to regard as inconsistent with one another. The apparent discrepancy may be cleared up as follows :—

Let

$$p=\mathrm{R}\frac{\rho}{\mu}\mathrm{T}, \qquad q=\tfrac{1}{3}a\mathrm{T}^4,$$

$$d(p+q)=-g\rho dr, \qquad d(p+q)=\frac{4\pi c\mathrm{GM}}{\mathrm{L}}\frac{1}{\eta k}dq$$

where p=gas pressure, q=radiation pressure, μ=mean molecular weight, M=mass of star, L=total radiation of star, k=absorption coefficient, η=ratio of L/M for the part of the star inside a given radius to L/M for the star as a whole, and the other notation is familiar.

Assume, following Jeans, that

$$k\propto\frac{\rho}{\mu\mathrm{T}^{3+n}},$$

$$\eta\propto\mathrm{T}^l, \qquad \frac{\rho}{\mu}\propto\rho^\phi, \qquad \phi=1+\psi,$$

n, l, ψ are constants.

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* *M.N.*, **85**, 196, 1925. † *M.N.*, **85**, 403, 1925.

(1) For a star of great mass p/q is very small, so that

$$k\eta = 4\pi c\mathrm{G}\frac{\mathrm{M}}{\mathrm{L}},$$

and is sensibly constant. But

$$k\eta \propto \frac{\rho}{\mu}\mathrm{T}^{l-n-3} \propto \rho^{1+\psi}\mathrm{T}^{l-n-3}.$$

The total pressure P is sensibly equal to q, and

$$\mathrm{P} = \tfrac{1}{3}a\mathrm{T}^4 \propto \rho^{(4+4\psi)/(3+n-l)}$$

throughout any given star. Hence the equilibrium is polytropic, and different stars are homologous, whatever the value of M or of the radius r (so long as M is great enough). At homologous points η will be the same: whence

$$\mathrm{L} \propto \frac{\mathrm{M}}{k} \propto \mathrm{M}\mathrm{T}^{3+n}\rho^{-1-\psi}a\mathrm{M}^{-\psi}\mathrm{T}^{3+n}r^{3+3\psi}.$$

But $\mathrm{T} \propto \mathrm{M}^{\frac{1}{3}}r^{-1}$ (since $\mathrm{P} \propto \mathrm{T}^4 a\mathrm{M}^2 r^{-4}$).
Hence, taking logarithms,

$$2\log \mathrm{L} = (3+n-2\psi)\log \mathrm{M} + (6\psi - 2n)\log r + \text{const.}$$

But if T_e is the effective surface temperature,

$$2\log r = \log \mathrm{L} - 4\log \mathrm{T}_e + \text{const.},$$

whence

$$\log \mathrm{L} = \frac{3+n-2\psi}{2+n-3\psi}\log \mathrm{M} + \frac{4n-12\psi}{2+n-3\psi}\log \mathrm{T}_e + \text{const.} \quad . \quad (\mathrm{A})$$

When $\psi = 0$ (constant molecular weight) this becomes identical with Eddington's equation (9a) (*M.N.*, **85**, 406) which was derived by similar reasoning.

Jeans' equations (*M.N.*, **85**, 206), when the mass is great (λ small), and $l = 0$, are

$$(4s+9)\log \mathrm{M} = 6\log \mathrm{E} + (4s+3)\left(\frac{3n-4}{2n-2}\right)\log f. \quad . \quad (51)$$

$$4\log \mathrm{T} + \tfrac{2}{3}\log \mathrm{M} - \log \mathrm{E} + \frac{2}{3(n-1)}\log f + \text{const.} \quad . \quad (52)$$

Eliminating f, these give

$$3\mathrm{Q}'\log \mathrm{T} + (\tfrac{1}{2}\mathrm{Q}' - 4s - 9)\log \mathrm{M} + (6 - \tfrac{3}{4}\mathrm{Q}')\log \mathrm{E} = \text{const.}$$

where

$$\mathrm{Q}' = (4s+3)(3n-4).$$

Now in our present notation

$$\mathrm{E} = \mathrm{L}\,;\ \mathrm{T} = \mathrm{T}_e\,;\ s = \tfrac{1}{4}(n-l) = \tfrac{1}{4}n,$$

$$n = \frac{4\phi}{4s+3} = \frac{4+4\psi}{n+3},$$

so that

$$\mathrm{Q}' = 12\psi - 4n,$$

and Jeans' equation becomes

$$(2+n-3\psi)\log \mathrm{L}=(3+n-2\psi)\log \mathrm{M}+(4n-12\psi)\log \mathrm{T}_e+\text{const.},$$

agreeing exactly with (A).

(2) For a star of very small mass, q/p is negligible.

Following Eddington's argument again,

$$dp=\frac{4\pi c\mathrm{GM}}{\mathrm{L}}\frac{1}{\eta k}\,dq,$$

whence

$$dp\propto \rho^{-1-\psi}\mathrm{T}^{3+n-l}d(\mathrm{T}^4)\propto \rho^{-1-\psi}\mathrm{T}^{6+n-l}d\mathrm{T}.$$

Now

$$p\propto \frac{\rho}{\mu}\mathrm{T}\propto \rho^{1+\psi}\mathrm{T}.$$

$$\therefore\ pdp\propto \mathrm{T}^{7+n-l}d\mathrm{T},$$

whence

$$p\propto \mathrm{T}^{4+\frac{1}{2}n-\frac{1}{2}l},$$

the additive constant which appears on integration being negligible except near the surface.

Hence

$$\rho^{1+\psi}\propto \mathrm{T}^{3+\frac{1}{2}n-\frac{1}{2}l},$$

and the equilibrium is again polytropic, and all stars of sufficiently small mass are homologous.

Here, however,

$$\frac{\mathrm{L}}{\mathrm{M}}\propto \frac{1}{\eta k}\frac{dq}{dp},$$

η is the same at homologous points, but k and $\frac{dq}{dp}$ are different.

On account of the homology, however,

$$\frac{dq}{dp}=\frac{q}{p}.$$

Hence

$$\mathrm{L}\propto \frac{\mathrm{M}}{k}\frac{q}{p}\propto \frac{\mathrm{M}}{\frac{\rho}{\mu}\mathrm{T}^{3-n}}\frac{\mathrm{T}^4}{\mathrm{M}^2r^{-4}},$$

or

$$\mathrm{L}\propto \mathrm{M}^{-1}\rho^{-1-\psi}\mathrm{T}^{7+n}r^{4\psi}\propto \mathrm{M}^{-2-\psi}\mathrm{T}^{7+n}r^{7+3\psi}.$$

Now at the centre

$$\mathrm{T}\propto \frac{\mu p}{\rho}\propto \mu\frac{\mathrm{M}}{r}\propto \rho^{-\psi}\frac{\mathrm{M}}{r}\propto \mathrm{M}^{1-\psi}r^{3\psi-1}.$$

Hence

$$2\log \mathrm{L}=(10+2n-16\psi-2n\psi)\log \mathrm{M}+(48\psi+6n\psi-2n)\log r+\text{const.}$$

Now

$$\log r=\tfrac{1}{2}\log \mathrm{L}-2\log \mathrm{T}_e-\text{const.},$$

whence

$$(2+n-3n\psi-24\psi)\log L=(10+2n-16\psi-2n\psi)\log M +4(n-3n\psi-24\psi)\log T_e+\text{const.} \quad . \quad . \quad (B)$$

When $\psi=0$ this becomes

$$\log L=\frac{2(5+n)}{2+n}\log M+\frac{4n}{2+n}\log T_e+\text{const.},$$

identical with Eddington's (9*b*).

Jeans' equation (53) is

$$3Q\log T+(\tfrac{1}{2}Q-2s-\tfrac{1}{4}^5)\log M+\tfrac{3}{4}(1-Q)\log E=\text{const.}$$

where

$$Q=12(\phi-1)+(6\phi-8)s.$$

In our notation these become

$$2Q=24\psi+3n\psi-n$$
$$(96\psi+12n\psi-4n)\log T_e+(16\psi+2n\psi-2n-10)\log M +(2+n-3n\psi-24\psi)\log L=\text{const.},$$

thus reproducing (B).

The equations (A) and (B) therefore include both Eddington's and Jeans' equations as special cases—the former when $\psi=0$ and the latter when $l=0$.

It is worthy of note that

$$\frac{dL}{d\log T_e}=\frac{4n-12\psi}{2+n-3\psi}$$

for great masses, and

$$\frac{dL}{d\log T_e}=\frac{4n-12(n+8)\psi}{2+n-3(n+8)\psi}$$

for small masses.

This quantity will be negative for great masses if $\psi>\frac{1}{3}n$, and for small masses if $\psi>\frac{1}{3}\frac{n}{n+8}$.

It is very probable that n is of the order of $\frac{1}{2}$, while ψ is probably of the order of $\frac{1}{50}$ (Jeans, *M.N.*, **85**, 399, puts it between 0·01 and 0·04).

It follows that $\frac{dL}{dT_e}$ is almost certainly positive for massive stars (so long as the assumptions here made can be trusted), while for stars of small mass it may be zero or perhaps negative.

Now Eddington has shown (*M.N.*, **85**, 405) that a star of fixed mass will be capable of automatic and stable adjustment of its radius so that its luminosity may have any assigned value when, and only when, $\frac{dL}{dT_e}$ is negative. His conclusion that such an adjustment cannot actually occur, and that the rate of internal generation of heat must rather be adjusted to balance the loss at the surface, appears to be fully verified, in the more general case, for massive stars, but perhaps not for those of small mass.

It should be borne in mind, however, that (as Eddington has pointed

out) the assumption $\eta \alpha T^l$ leads to the very improbable conclusion that a large part of the energy generated in the inner portion of the star is continuously absorbed in the outer layers. More probable assumptions would destroy the polytropic character of the equilibrium, and complicate the analysis greatly. The formulæ (A) and (B) (and the others recently obtained by Eddington and Jeans) may therefore be expected to be qualitative, rather than exactly quantitative, approximations to the truth. That they are good approximations, nevertheless, is shown by their agreement with the observed data.

Applications of Statistical Mechanics to determine the Properties of Matter in Stellar Interiors. Part I. *The Mean Molecular Weight.* By R. H. Fowler and E. A. Guggenheim.

§ 1. *Introduction.*—The contributions which statistical mechanics can make to Eddington's well-known theories * of the equilibrium conditions in the interior of a gaseous star are concerned with the properties of the material of which the star is composed. In the first place, it can attempt to determine as accurately as possible the degree of ionisation of any kind of atom at the temperatures and densities assigned by the theory to the various levels in a star, and thus the mean molecular weight of the stellar material, so that crude approximations to this important magnitude can be avoided. Secondly, it can calculate the gas-pressure as a function of the temperature and density, in particular studying its departures (so long at least as they are small) from the pressure of a perfect gas. Thirdly, it can attempt to assign the manner in which atomic ions in any given stage of ionisation are distributed among their possible stationary states. This will be required in the computation of the mean mass absorption coefficient when the latter is better known for the individual stationary states. Lastly, it can be used to fix the form of the (P, V)-adiabatic curves of the stellar material, which plays an important part in Eddington's theory of pulsating stars (Cepheid variables).†

In the present paper we apply the equations of dissociative equilibrium, including corrections for the excited states, excluded volumes, and electrostatic charges of the atomic ions, to the first two of these problems, studying therefore the gas-pressure as a function of the temperatures and densities which are supposed to occur in giant and dwarf stars of various compositions. The corrections to the perfect gas laws have recently become of exceptional interest in this connection, since Eddington has pointed out that his theory, originally promulgated for diffuse giant stars, may be expected to hold even for dwarfs, in which the central densities must be many times that of water. He points out that, owing to the high degree of ionisation to be expected, the remaining

* Eddington, *Zeit. f. Phys.*, **7**, 351 (1921), systematising previous papers in *M.N.*; also *M.N.*, **83**, 32, 98, 431 (1923); **84**, 104, 308 (1924).

† Eddington, *M.N.*, **79**, 2, 177 (1919).

atomic cores will be very small compared with normal atoms, so that the perfect gas laws may persist to these densities. This point we examine in detail, and find only comparatively small deviations up to densities as high as 400 at the high temperatures expected to accompany such densities in stellar interiors. In giant stars the deviations from the perfect gas laws seldom reach 5 per cent. We find also that throughout the important range of temperatures and densities the mean molecular weight is almost constant for matter of a given composition; if the stars were mostly oxygen the value would be about 1·9 to 2·0, if silver 2·4 to 2·9. By assuming that iron is the element most representative of the mean composition of the stellar interior, the mean molecular weight comes out at from 2·2 to 2·4. Eddington's most recent assumed values are 2·11 and 2·2. The presence of very much hydrogen would lower this value towards 2·0, but no reasonable admixture of hydrogen has any appreciable effect.

In a second paper we shall apply our present results to the approximate determination of the (P, V)-adiabatics for use in Eddington's theory of the Cepheid variables.

We do not in this paper consider the problem of the radial distribution of different sorts of atoms. This is a distinct problem to which Rosseland has made a preliminary contribution.* Nor do we attempt to recalculate any of Eddington's values of temperatures and densities for particular stars, for the corrected values of the mean molecular weight here derived. The changes would be small and as yet barely significant. The new values can always be regarded as the correct values for a star of mass and radius slightly different from the best values for the particular star in question. What is aimed at here is to give a general account of the nature of stellar material so far as this can yet be determined in view of the unavoidable uncertainties that still remain in the physical theory. These uncertainties, though some of them are of theoretical importance, probably do not seriously influence the results presented in this paper. We have tried in presenting our results to avoid implicating them in astronomical and astrophysical uncertainties. We have therefore in the main presented all our calculations as functions of assigned temperatures and densities, and have only correlated them in a rough-and-ready way with particular stars.

We make a categorical statement of the theoretical formulæ on which the calculations here given are based. For the sake of the calculations some one form had to be adopted, which is unlikely to prove ultimately entirely correct. It would make this paper too long to examine in detail the uncertainties in the formulæ and the directions in which they may be modified in the future. As we have just stated, we believe that the astrophysical importance of these uncertainties and modifications is now small. Those concerned, however, with such questions will find them discussed by one of us in two papers, which will be published shortly.† The formulæ there given must be taken entirely to supersede the statistical formulæ used in the work of Fowler and Milne in this publication.

* Rosseland, *M.N.*, **84**, 720 (1924).

† Fowler, *Proc. Camb. Phil. Soc.* (1925); *Phil. Mag.* (in publication).

The work, here presented in an amplified form, was part of an essay on Statistical Mechanics, with special reference to the physical state of matter at high temperatures, which was awarded the Adams Prize for 1923–24.

§ 2. *Method of Calculation of the Equilibrium State of the Stellar Material.*—The notation used is the following :—

m, the mass of an electron, $9{\cdot}00 \times 10^{-28}$ gm.
h, Planck's constant, $6{\cdot}55 \times 10^{-27}$ c.g.s.u.
k, Boltzmann's constant, $1{\cdot}372 \times 10^{-16}$ ergs/°C.
R, Rydberg's constant, $1{\cdot}097 \times 10^{5}$ cm.$^{-1}$.
ϵ, the charge on the electron, $4{\cdot}774 \times 10^{-10}$ e.s.u.
T, absolute temperature in centigrade degrees (°K).
ρ, density, gm./c.c.
V, volume of the system under discussion, in general unity (1 c.c.).
N, number of *free* electrons in volume V.
M_r, number of atoms of a given element in volume V that have lost r electrons.
$M(=\Sigma_r M_r)$, total number of atoms of the given element in volume V.
χ_r, the work required to remove to infinity the $(r+1)^{\text{th}}$ electron from an atomic ion, that has already lost r electrons, in its normal state in free space.
$s\ (=N/M)$, the mean number of electrons lost by atoms of given type.
A, the atomic weight of an element.
μ, the mean molecular weight of the material.

The general equations for dissociative equilibrium in a form suitable to the discussion of highly ionised gases have been discussed by one of us in two recent papers.* The results of this discussion can only be quoted here and form the starting-point for the present calculations. The principal formula is

$$\frac{NM_{r+1}}{VM_r} = \frac{(2\pi mkT)^{\frac{3}{2}}}{h^3} \frac{u_{r+1}(T)}{u_r(T)} e^{-\chi_r/kT - J + L} \quad . \quad . \quad (1)$$

In equation (1) J represents a correction for excluded volumes and is shown (*loc. cit.*) to be less than 1/30. It is therefore always ignored here, the resulting error being far less than the unavoidable uncertainties of the present calculations. L represents the correction for the electrostatic charges of the ions, which can be calculated by the theory of Debye and Hückel,† and is given by

$$L = \frac{2\sqrt{\pi\epsilon^3}}{(kT)^{\frac{3}{2}}} \left(\frac{N}{V}\right)^{\frac{1}{2}} (r+1) \left(1 + \frac{1}{N}\Sigma_r r^2 M_r\right)^{\frac{1}{2}} \quad . \quad . \quad (2.1)$$

or with sufficient accuracy ‡ by

$$L = \frac{2\sqrt{\pi\epsilon^3}}{(kT)^{\frac{3}{2}}} \left(\frac{N}{V}\right)^{\frac{1}{2}} (r+1)(s+1)^{\frac{1}{2}} \quad . \quad . \quad . \quad (2.2)$$

* Fowler, *loc. cit.*

† Debye and Hückel, *Phys. Zeit.*, **24**, 185, 305 (1923). The statistical basis is discussed further by Fowler, *loc. cit.*

‡ Strictly the last factor is $(1+\overline{r^2}/s)^{\frac{1}{2}}$ and we have identified $\overline{r^2}$ and $(\bar{r})^2$ $(=s^2)$.

The factors $u_r(\mathrm{T})$ and $u_{r+1}(\mathrm{T})$ take account of the fact that the atomic ions of any stage of ionisation must be distributed among their various stationary states, excited ones as well as normal. The function $u_r(\mathrm{T})$ is

$$u_r(\mathrm{T})=\sum_{s=0}^{\infty}(\varpi_r)_s e^{-\{\chi_r-(\chi_r)_s\}/k\mathrm{T}-\mathrm{N}v_s/\mathrm{V}} \quad . \quad . \quad (3)$$

The quantity v_s is an atomic excluded volume, so that $\mathrm{N}v_s/\mathrm{V}$ is a term of the type b in van der Waals' equation; $(\varpi_r)_s$ is the weight of the s^{th} state of the r-times ionised atom, and $(\chi_r)_s$ is the energy required to remove to infinity the electron from this s^{th} state. The state $s=0$ is the normal state, so that $(\chi_r)_0=\chi_r$.

The following approximation is used for $u_r(\mathrm{T})$, for details of which the reader must refer to the papers quoted :—

$$u_r(\mathrm{T})=(\varpi_r)_0+\left[\frac{1}{3}\Gamma\left(\frac{5}{3}\right)\left(\frac{3}{4}\right)^{\frac{5}{3}}\frac{\mathrm{R}hck}{\pi\epsilon^4}(r+1)^2(\Sigma_i 2i)\right]\frac{\mathrm{T}e^{-\chi_r/k\mathrm{T}}}{(\mathrm{N}/\mathrm{V})^{\frac{2}{3}}} \quad (4)$$

In (4) $(\varpi_r)_0$ is the weight of the normal term, all terms of a multiple term being lumped together; it is tabulated below for the atomic types of importance here. $\Sigma_i(2i)$ is the sum of the maximum multiplicities of all sets of stationary states (spectral terms) which the atomic ion possesses. The values used here are tabulated below, but they are somewhat uncertain. They should perhaps be considerably increased, but, even so, such increases would not introduce any changes of importance in our results. In general one of the terms in (4) is very small compared to the other. For the largest values of r the second term is small, and we have simply

$$u_r(\mathrm{T})=(\varpi_r)_0 \quad . \quad . \quad . \quad . \quad (5.1)$$

and in particular for bare nuclei

$$u_z(\mathrm{T})=1 \quad . \quad . \quad . \quad . \quad (5.2)$$

where z is the atomic number. For low values of r, on the other hand, the second term is large, and

$$u_r(\mathrm{T}) \text{ const.}\times(r+1)^2(\Sigma_i 2i)\frac{\mathrm{T}e^{-\chi_r/k\mathrm{T}}}{(\mathrm{N}/\mathrm{V})^{\frac{2}{3}}} \quad . \quad . \quad (5.2^1)$$

so that

$$\frac{u_{r+1}(\mathrm{T})}{u_r(\mathrm{T})}=\frac{e^{-\chi_{r+1}/k\mathrm{T}}}{e^{-\chi_r/k\mathrm{T}}}\left(\frac{r+2}{r+1}\right)^2\frac{(\Sigma_i 2i)_{r+1}}{(\Sigma_i 2i)_r} \quad . \quad . \quad (5.3)$$

It is seldom that formulæ other than either (5.1) or (5.3) have to be used, and in these the remaining uncertainties are fortunately small.

Obviously the most important atomic constants for which we require numerical values are the χ_r. The values used here for Fe, O, Ag are those specially computed for this purpose by Hartree,* and we have computed others by his methods. They are the energies required for the successive removal of electrons, and it is far from sufficiently accurate to use the energy values of the corresponding X-ray levels of

* Hartree, *Proc. Camb. Phil. Soc.*, **22**, 464 (1924).

the normal atom. We could have made no reasonably secure progress without Hartree's work.

The remaining constants ϖ_0 and $\Sigma_i 2i$ for ions of different types are given in the following table:—

List of Atomic Constants used in these Calculations.

[Values not guaranteed as final; some are certainly wrong.]

Typical Element.	$z-r$.	ϖ_0.	$\Sigma 2i$.
Bare nucleus.	0	1	0
H	1	2	1
He	2	1	1+1+1 = 3
Li	3	2	2
Be	4	1	1+3 = 4
B	5	6	2
C	6	3	1+3 = 4
N	7	2	2+4 = 6
O	8	9	1+3+5 = 9
F	9	4	2+4+6 =12
Ne	10	1	1+3+3+5=12
Na	11	2	2
Mg	12	1	1+3 = 4
Al	13	6	2
Si	14	3	1+3 = 4
P	15	2	2+4 = 6
S	16	9	1+3+5 = 9

These values are of course derived from spectral analysis and are used on the assumption that the structure of all ions with a given number of electrons is the same whatever the nuclear charge. They are not all equally certain, and some are certainly wrong. But the most important ones for the following work are the most certain, and the remaining errors and uncertainties will have little if any appreciable effect on the derived mean molecular weight.*

To determine the state of dissociative equilibrium of a given element for a given temperature and density we start by guessing a value of N/V (or N if V=1); this is equivalent to guessing s, the average degree of dissociation. With the help of this guess the value of the electrostatic correction term L can be approximately found (to be verified a posteriori), and then the ratios M_{r+1}/M_r can be exactly calculated. Let us write

$$M_{r+1}/M_r = f_r \quad . \quad . \quad . \quad . \quad (6)$$

then

$$f_r = 2{\cdot}435\times10^{15}\frac{T^{\frac{3}{2}}}{N/V}\,\frac{u_{r+1}(T)}{u_r(T)}e^{-\chi_r/kT+L} \quad . \quad . \quad (7)$$

* More recent values, probably nearly correct, have lately become available and are tabulated by Fowler, *loc. cit.*

L is always small compared with χ_r/kT, sometimes even negligible. For the ordinary values of T and N/V encountered f_r is large so long as $e^{-\chi_r/kT}$ is not very small, and under the same conditions the values of $u_r(T)$ are controlled by the second term in (4), so that the ratio u_{r+1}/u_r is given by (5.3). Thus we see, as is of course to be expected, that f_0 and all the other early f's are in general large, but that there is a steady decrease as r increases due to the increase of χ_r. In general therefore there comes a point where the f's are of the order unity ; after that they still continue to decrease steadily. Let t be the smallest value of r for which $f_r<1$; that is to say, suppose

$$f_{t-1}\geqslant 1, \quad f_t<1.$$

If all the f's are greater than unity we take $t=z-1$. Then we can write

$$M_{t+1}/M_t=f_t,\ M_{t+2}/M_t=f_{t+1}f_t,\ \ldots$$
$$M_{t-1}/M_t=1/f_{t-1},\ M_{t-2}/M_t=1/f_{t-1}f_{t-2},\ \ldots$$

so that

$$\frac{M}{M_t}=1+f_t+f_{t+1}f_t+\ \ldots\ +\frac{1}{f_{t-1}}+\frac{1}{f_{t-2}f_{t-1}}+\ \ldots, \quad . \qquad (8)$$

and if there is only one element present

$$\frac{N-tM}{M_t}=f_t+2f_{t+1}f_t+3f_{t+2}f_{t+1}f_t+\ \ldots-\frac{1}{f_{t-1}}-\frac{2}{f_{t-2}f_{t-1}}-\frac{3}{f_{t-3}f_{t-2}f_{t-1}}-\ldots \qquad (9)$$

From (8) and (9) we calculate at once $(N-tM)/M$, and so

$$s=\frac{N}{M}=t+\frac{N-tM}{M} \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The mean molecular weight follows at once from

$$\mu=A/(1+s) \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

In the four series occurring in (8) and (9), the terms become rapidly smaller, so that all but the first few are negligible. It is therefore only necessary to evaluate a few of the f's, generally less than six. A similar method can be used for material consisting of any number of elements. There is an equation equivalent to (8) for each element. In place of the combination of (8) and (9), $N-tM=qM$, we have an equation

$$N-\Sigma_A tM=\Sigma_A qM \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

where Σ_A is a summation over all the elements present. For assumed ratios between the M's and the assumed N (12) determines the M's and so the mean molecular weight. When s (or the values of s for each atom) has been thus calculated, the value of the electrostatic term L is verified and, if necessary, adjusted. Finally, the density ρ is calculated ; if the value obtained differs from that required, the calculations are repeated with a different value of N.

§ 3. *Results of Calculations.*—The first set of calculations, given in Table I., is for the temperatures and densities at various central distances in a typical giant star, mass $1\frac{1}{2}$ times the Sun's mass.* We

* Eddington, *Zeit. f. Phys., loc. cit.*, p. 375.

have chosen Fe, O, Ag with certain admixtures of H as typical constitutions, which are specified in percentages of atoms present (not percentages of masses).

TABLE I.

Equilibrium States of Matter at the Temperatures and Densities in Eddington's Typical Giant $M^*/\odot = 1{\cdot}5$.

T °K. [Central Distance.]	ρ. gm./c.c.	Composition assumed in Percentages of Atoms.	Free Electrons per c.c. N.	Free Electrons per Atom s.	Mean Molecular Weight μ.	Chief States of Ionisation in % of Total Atoms.
$6{\cdot}59\times10^6$	0·1085	100 % Fe	$2{\cdot}70\times10^{22}$	22·98	2·33	M_{23}, 94.
[0]		100 % O	$2{\cdot}97\times10^{22}$	7·23	1·95	M_8, 23; M_7, 76.
		100 % Ag	$2{\cdot}23\times10^{22}$	36·57	2·87	M_{37}, 46; M_{36}, 45.
		50 % Fe, 50 % O	$2{\cdot}77\times10^{22}$	22·98 7·24	2·23	
		50 % Fe, 50 % Ag	$2{\cdot}39\times10^{22}$	22·98 36·53	2·66	
		50 % O, 50 % Ag	$2{\cdot}33\times10^{22}$	7·26 36·54	2·70	
		50 % Fe, 50 % H	$2{\cdot}78\times10^{22}$	22·98	2·18	
		50 % O, 50 % H	$3{\cdot}17\times10^{22}$	7·22	1·66	
		50 % Ag, 50 % H	$2{\cdot}36\times10^{22}$	36·54	2·75	
$4{\cdot}64\times10^6$	0·0678	100 % Fe	$1{\cdot}69\times10^{22}$	22·97	2·33	M_{23}, 93.
$[r_0/6{\cdot}9]$						
$2{\cdot}84\times10^6$	0·0215	100 % Fe	$5{\cdot}33\times10^{21}$	22·82	2·34	M_{23}, 78.
$[2r_0/6{\cdot}9]$		100 % O	$5{\cdot}95\times10^{21}$	7·28	1·93	M_8, 29; M_7, 70.
		100 % Ag	$4{\cdot}30\times10^{21}$	35·63	2·94	M_{37}, 8; M_{36}, 61; M_{35}, 28.
$1{\cdot}37\times10^6$	0·0050	100 % Fe	$1{\cdot}135\times10^{21}$	21·03	2·53	M_{22}, 34; M_{21}, 22; M_{20}, 22.
$[3r_0/6{\cdot}9]$		100 % O	$1{\cdot}39\times10^{21}$	7·32	1·92	M_8, 34; M_7, 64.
		100 % Ag	$8{\cdot}68\times10^{20}$	31·00	3·37	M_{32}, 21; M_{31}, 32; M_{30}, 26.
$0{\cdot}838\times10^6$	0·0010	100 % Fe	$1{\cdot}86\times10^{20}$	16·67	3·16	M_{18}, 12; M_{17}, 46; M_{16}, 38.
$[4r_0/6{\cdot}9]$		100 % O	$2{\cdot}74\times10^{20}$	7·21	1·95	M_8, 33; M_7, 55; M_6, 11.
		* 100 % Ag	$1{\cdot}47\times10^{20}$	26	4·0	

The features of this table which may be emphasised are (*a*) the very slow radial variation of μ for constant radial composition. With any concentration of heavier elements to the centre this variation of μ would be still further reduced.

(*b*) The very small effect of mixtures on the degree of ionisation of a single element at given temperature and density.

* These values are very rough, as the important silver ions have the same number of electrons as elements of the iron transition group, so that their electronic configurations could not be deduced with certainty from existing spectroscopic data.

(*c*) The excessive amount of H which would be required to have any serious effect on the mean molecular weight.

(*d*) The most probable value of the mean molecular weight for Capella is about 2·33. It cannot well be so small as 2·1 or even 2·2, unless Capella is composed of elements on the average substantially lighter than Fe.

Table II. gives similar calculations for the central conditions in a series of Eddington's theoretical stars in which the radius of the star is varied. The theoretical conditions at the centre satisfy the equations*

$$T_c = 6{\cdot}84\times10^{-16}\mu\beta M^*/r_0, \quad . \quad . \quad . \quad (13.1)$$

where M^* is the mass of the star and r_0 its radius (in c.g.s. units),

$$\rho_c = 12{\cdot}95 M^*/r_0^3, \quad . \quad . \quad . \quad . \quad (13.2)$$
$$1-\beta = 0{\cdot}00309\mu^4\beta^4(M^*/\odot)^2, \quad . \quad . \quad . \quad (13.3)$$

where $\odot$ is the mass of the Sun. We have taken the central temperature and density for the typical giant of Table I. and repeatedly divided r_0 by 2, ignoring other changes. This gives the series of temperatures and densities assumed in Table II. for which the equilibrium state of the matter has been calculated.

TABLE II.

Equilibrium States of Matter for the Centres of Stars of the Mass $M^*/\odot = 1{\cdot}5$ *and Various Radii.*

R/R_c is the ratio of the assumed radius to the radius of the standard giant of Table I.

T °K. [R/R$_c$.]	ρ. gm./c.c.	Composition assumed.	Free Electrons per c.c. N.	Free Electrons per Atom *s*.	Mean Molecular Weight μ.	Chief States of Ionisation in % of Total Atoms.	Remarks
$6{\cdot}59\times10^6$	0·1085	Fe	$2{\cdot}70\times10^{22}$	22·98	2·33	M_{23}, 94.	
[1]		O	$2{\cdot}97\times10^{22}$	7·23	1·95	M_8, 23; M_7, 76.	
		Ag	$2{\cdot}23\times10^{22}$	36·57	2·87	M_{37}, 46; M_{36}, 45.	
$1{\cdot}318\times10^7$	0·867	Fe	$2{\cdot}20\times10^{23}$	23·40	2·28	M_{24}, 35; M_{23}, 61.	
[$\frac{1}{2}$]		O	$2{\cdot}36\times10^{23}$	7·18	1·96	M_8, 19; M_7, 80.	
		Ag	$2{\cdot}11\times10^{23}$	43·29	2·44	M_{44}, 48; M_{43}, 40.	
$2{\cdot}636\times10^7$	6·95	Fe	$1{\cdot}85\times10^{24}$	24·56	2·18	M_{25}, 57; M_{24}, 37.	Roughl
[$\frac{1}{4}$]		O	$1{\cdot}87\times10^{24}$	7·13	1·97	M_8, 14; M_7, 84.	average c
		Ag	$1{\cdot}71\times10^{24}$	43·90	2·40	M_{44}, 89; M_{43}, 10.	ditions
							the Sur
$5{\cdot}27\times10^7$	55·4	Fe	$1{\cdot}49\times10^{25}$	24·80	2·16	M_{25}, 79; M_{24}, 19.	
[$\frac{1}{8}$]		O	$1{\cdot}49\times10^{25}$	7·065	1·98	M_8, 10; M_7, 86.	
		Ag	$1{\cdot}37\times10^{26}$	44·22	2·39	M_{45}, 26; M_{44}, 69.	
$1{\cdot}054\times10^8$	444	Fe	$1{\cdot}195\times10^{26}$	24·79	2·16	M_{25}, 78; M_{24}, 20.	
[$\frac{1}{16}$]		O	$1{\cdot}17\times10^{26}$	6·98	2·01	M_8, 7; M_7, 84; M_6, 9.	
		Ag	$1{\cdot}115\times10^{26}$	44·73	2·36	M_{45}, 74; M_{44}, 23.	

* Eddington, *loc. cit.*

There are no startling features in this table. The ionisation of O is slowly decreasing, that of Fe has just reached its maximum along this sequence of temperatures and densities. It will be observed that in virtue of (7) any element should have a maximum degree of ionisation somewhere on the sequence, for $T^{\frac{3}{2}}V/N \propto r_0^3$, and when T has become sufficiently large all exponential factors such as $e^{-\chi_r/kT}$ are unity. Thus the ionisation must ultimately decrease. There is a rapid increase in the ionisation of Ag in the first step, due to the removal of the greater part of its L-electrons. After that its ionisation increases only slowly. The third row should correspond roughly to mean conditions in the Sun.*

It may be observed here that the calculations of Tables I. and II., in so far as they lead to different values of μ to those assumed by Eddington and only small deviations from the perfect gas laws, merely refer to stars of slightly different mass, radius, and β to those of the star from which we started. If μ is assigned the correct mean value, then equations (13) can be solved for M*, r_0, and β for the selected T_c and ρ_c and the derived μ. The T_c, ρ_c, and β are then characteristic of the slightly varied star.

Table III. gives results for sets of temperatures and densities which, according to Eddington's latest estimates,† correspond to central and average conditions in four typical Cepheids ; we have followed Eddington's suggestion that the average condition should be represented by $\frac{2}{3}$ central temperature and $(\frac{2}{3})^3$ central density. This table includes also calculations for mean conditions in two standard stars, *g* and *f* (p. 948).

TABLE III.

Equilibrium States of Matter at the Temperatures and Densities in Four Typical Cepheids.

Star.	T °K.	ρ. gm./c.c.	Composition.	Free Electrons per c.c. N.	Free Electrons per Atom *s*.	Mean Molecular Weight μ.	Chief States of Ionisation in % of Total Atoms.
Oph. central	$4{\cdot}24\times10^6$	0·0029	Fe	$7{\cdot}24\times10^{20}$	23·03	2·32	M_{24}, 5 ; M_{23}, 94.
			O	$8{\cdot}80\times10^{20}$	7·47	1·89	M_8, 47 ; M_7, 53.
			Ag	$5{\cdot}89\times10^{20}$	36·14	2·91	M_{37}, 14 ; M_{36}, 86.
Oph. average	$2{\cdot}82\times10^6$	0·0009	Fe	$2{\cdot}24\times10^{20}$	22·94	2·33	M_{23}, 84 ; M_{22}, 11.
			O	$2{\cdot}55\times10^{20}$	7·50	1·88	M_8, 50 ; M_7, 50.
			Ag	$1{\cdot}805\times10^{20}$	35·70	2·94	M_{36}, 69 ; M_{35}, 27.
Aqu. central	$6{\cdot}00\times10^6$	0·015	Fe	$3{\cdot}74\times10^{21}$	23·02	2·32	M_{23}, 95.
			O	$4{\cdot}18\times10^{21}$	7·37	1·92	M_8, 37·5 ; M_7, 62·5.
			Ag	$3{\cdot}13\times10^{21}$	37·10	2·83	M_{38}, 20; M_{37}, 57; M_{36}, 19.
Aqu. average	$4{\cdot}00\times10^6$	0·0047	Fe	$1{\cdot}17\times10^{21}$	23·01	2·32	M_{23}, 93.
			O	$1{\cdot}32\times10^{21}$	7·41	1·90	M_8, 41 ; M_7, 59.
			Ag	$9{\cdot}54\times10^{20}$	36·05	2·91	M_{37}, 5 ; M_{36}, 94·5.

* Eddington, *M.N.*, **84**, 324 (1924), table vii.

† By letter.

TABLE III—*continued.*

Star.	T °K.	ρ. gm./c.c.	Composition.	Free Electrons per c.c. N.	Free Electrons per Atom s.	Mean Molecular Weight μ.	Chief States of Ionisation % of Total Atoms.
RT Aur. central	$8{\cdot}76\times10^{6}$	0·066	Fe	$1{\cdot}66\times10^{22}$	23·23	2·30	M_{24}, 22 ; M_{23}, 77.
			O	$1{\cdot}82\times10^{22}$	7·30	1·93	M_{8}, 31 ; M_{7}, 69.
			Ag	$1{\cdot}51\times10^{22}$	40·87	2·58	M_{43}, 18 ; M_{42}, 12 ; M_{41}, M_{40}, 20 ; M_{39}, 15.
RT Aur. average	$5{\cdot}85\times10^{6}$	0·0208	Fe	$5{\cdot}19\times10^{21}$	23·02	2·32	M_{23}, 96.
			O	$5{\cdot}78\times10^{21}$	7·34	1·92	M_{8}, 34·5 ; M_{7}, 65·5.
			Ag	$4{\cdot}31\times10^{21}$	36·82	2·85	M_{38}, 12 ; M_{37}, 55 ; M_{36},
RR Lyr. central	$1{\cdot}83\times10^{7}$	1·18	Fe	$3{\cdot}11\times10^{23}$	24·33	2·21	M_{25}, 41 ; M_{24}, 49 ; M_{23},
			O	$3{\cdot}21\times10^{23}$	7·19	1·96	M_{8}, 20 ; M_{7}, 80.
			Ag	$2{\cdot}91\times10^{23}$	43·86	2·40	M_{44}, 86 ; M_{43}, 13.
RR Lyr. average	$1{\cdot}22\times10^{7}$	0·372	Fe	$9{\cdot}46\times10^{22}$	23·43	2·28	M_{24}, 39 ; M_{23}, 58.
			O	$1{\cdot}016\times10^{23}$	7·21	1·95	M_{8}, 22·5 ; M_{7}, 76.
			Ag	$9{\cdot}04\times10^{22}$	43·31	2·43	M_{44}, 49 ; M_{43}, 40 ; M_{42},
g	$1{\cdot}53\times10^{7}$	0·155	Fe	$4{\cdot}14\times10^{22}$	27·55	2·18	M_{25}, 54 ; M_{24}, 39.
			O	$4{\cdot}28\times10^{22}$	7·30	1·93	M_{8}, 30 ; M_{7}, 70.
			Ag	$3{\cdot}82\times10^{22}$	43·93	2·40	M_{44}, 92 ; M_{43}, 7.
f	$2{\cdot}00\times10^{6}$	0·00033	Fe	$7{\cdot}88\times10^{19}$	21·98	2·43	M_{23}, 26 ; M_{22}, 52 ; M_{21},
			O	$1{\cdot}04\times10^{20}$	7·53	1·88	M_{8}, 53 ; M_{7}, 47.
			Ag	$6{\cdot}01\times10^{19}$	32·40	3·33	M_{35}, 8 ; M_{34}, 11 ; M_{33}, 2 M_{32}, 29 ; M_{31}, 19 ; M_{30},

In Table IV. we give calculations for the temperatures and densities of Table I. (and a few others) to show the variation with atomic number of the distribution of the atoms among their various stages of ionisation. These calculations are principally of interest in connection with our second paper on the (P, V)-adiabatics. But they serve here to show the speed at which a change of atomic number takes one out of a transition region in which a particular atom is spread over several stages of ionisation, so that its mean molecular weight is varying comparatively rapidly. There is practically no overlap in the transition regions for Fe and Br, and only a moderate overlap for Fe and Zn and for Zn and Br. In order to facilitate comparisons of the numbers of the different atoms in similar states, we have denoted these here by the symbol of the atom which they resemble. For the purpose of these calculations we have estimated the successive ionisation potentials of these elements by the methods given by Hartree.* These are given in Table V. for reference.

* *Loc. cit.*

TABLE IV.

Element.	T °K.	ρ. gm./c.c.	Free Electrons per c.c. N.	s.	μ.	Chief Types of Ions in Percentages.
Fe	$6{\cdot}59\times10^6$	0·1085	$2{\cdot}70\times10^{22}$	22·98	2·33	Li 94.
Zn	..	..	$2{\cdot}71\times10^{22}$	26·91	2·39	Li 88, Be 10.
Br	..	..	$2{\cdot}83\times10^{22}$	31·44	2·46	Li 58, Be 35.
Ag	..	..	$2{\cdot}23\times10^{22}$	36·57	2·87	Ne 46, Na 45.
Fe	$4{\cdot}00\times10^6$	0·0047	$1{\cdot}17\times10^{21}$	23·01	2·32	Li 93.
Zn	..	..	$1{\cdot}17\times10^{21}$	26·81	2·35	Li 76, Be 17.
Br	..	..	$1{\cdot}04\times10^{21}$	29·27	2·74	Be 21, B 19, C 28, N 19, O 9.
Fe	$3{\cdot}84\times10^6$	0·0215	$5{\cdot}33\times10^{21}$	22·82	2·34	Li 78, Be 19.
Zn	..	..	$5{\cdot}11\times10^{21}$	25·72	2·45	Li 25, Be 44, B 15, C 11.
Br	..	..	$4{\cdot}18\times10^{21}$	25·82	2·98	N 6, O 22, F 28, Ne 28, Na 14.
Fe	$2{\cdot}82\times10^6$	0·0009	$2{\cdot}24\times10^{20}$	22·94	2·33	Li 84, Be 11.
Zn	..	..	$2{\cdot}14\times10^{20}$	25·67	2·45	Li 20, Be 45, B 19, C 12.
Br	..	..	$1{\cdot}75\times10^{20}$	25·70	2·99	O 16, F 33, Ne 45.
Ag	..	..	$1{\cdot}81\times10^{20}$	35·70	2·94	Na 69, Mg 27.
Fe	$2{\cdot}37\times10^6$	0·0050	$1{\cdot}135\times10^{21}$	21·03	2·53	Be 34, B 22, C 22.
Zn	..	..	$9{\cdot}73\times10^{20}$	21·01	2·96	O 24, F 40, Ne 29.
Br	..	..	$9{\cdot}08\times10^{20}$	24·18	3·18	Ne 19, Na 77.
Ag	..	..	$8{\cdot}68\times10^{20}$	31·00	3·37	S 21, Cl 31, A 26.
Fe	$1{\cdot}38\times10^6$	0·0010	$1{\cdot}86\times10^{20}$	16·67	3·16	O 12, F 46, Ne 38.
Zn	..	..	$1{\cdot}77\times10^{20}$	19·08	3·25	Ne 11, Na 85.

TABLE V.

Successive Ionisation Potentials of Zn *and* Br.

Zn. N=30.	χ_r, Volts.	Br. N=35.	χ_r, Volts.
Zn XXX *	12,150	Br XXXV	16,600
Zn XXIX	11,600	Br XXXIV	16,000
Zn XXVIII	2,720	Br XXXIII	3,800
Zn XXVII	2,580	Br XXXII	3,620
Zn XXVI	2,400	Br XXXI	3,400
Zn XXV	2,240	Br XXX	3,200
Zn XXIV	2,120	Br XXIX	3,060
Zn XXIII	1,940	Br XXVIII	2,840
Zn XXII	1,820	Br XXVII	2,700
Zn XXI	1,715	Br XXVI	2,560
Zn XX	750	Br XXV	1,100
Zn XIX	661	Br XXIV	1,020
Zn XVIII	605	Br XXIII	947

* The notation is the same as in Hartree's paper ; the Roman numeral is equal to the core charge and so exceeds r by unity.

For a series of calculations at higher densities for less massive stars, guided by table vii. in Eddington's last paper,* we have taken the fixed temperature $2{\cdot}636\times10^7$ (middle entry of Table II.) and varied solely the density. The results are given in Table VI. It is remarkable how extremely small is the dependence of the mean molecular weight on the density. We have specially tested this insensitivity further in

TABLE VI.

High Densities in Stars of Small Mass.

$T=2{\cdot}636\times10^7$ °K.

Composition.	ρ.	Free Electrons per c.c.	s.	μ.	Chief States in Percentages.
Fe (1)	6·95	$1{\cdot}85\times10^{24}$	24·56	2·18	M_{25}, 57 ; M_{24}, 37.
,, (2)	15·2	$4{\cdot}0\times10^{24}$	24·30	2·21	M_{25}, 40 ; M_{24}, 48 ; M_{23}, 10.
,, (3)	38·4	$1{\cdot}0\times10^{25}$	23·98	2·24	M_{25}, 23 ; M_{24}, 53 ; M_{23}, 22.
,, (4)	71·6	$1{\cdot}85\times10^{25}$	23·78	2·25	M_{25}, 15 ; M_{24}, 50 ; M_{23}, 31.
,, (5)	156	$4{\cdot}0\times10^{25}$	23·63	2·26	M_{25}, 10 ; M_{24}, 44 ; M_{23}, 38 ; M_{22}, 6.
,, (6)	392	$1{\cdot}0\times10^{26}$	23·52	2·28	M_{25}, 9 ; M_{24}, 41·5 ; M_{23}, 41·5 ; M_{22}, 7·5.
O (1)	6·94	$1{\cdot}87\times10^{24}$	7·13	1·97	M_8, 14 ; M_7, 84.
,, (2)	18·6	$5{\cdot}0\times10^{24}$	7·07	1·98	M_8, 11 ; M_7, 85.
,, (3)	37·6	$1{\cdot}0\times10^{25}$	7·01	2·00	M_8, 9 ; M_7, 83 ; M_6, 8.
,, (5)	156	$4{\cdot}0\times10^{25}$	6·75	2·06	M_7, 70 ; M_6, 25.
,, (6)	408	$1{\cdot}0\times10^{26}$	6·48	2·14	M_7, 52 ; M_6, 37 ; M_5, 6.
Ag (1)	6·94	$1{\cdot}71\times10^{24}$	43·90	2·40	M_{44}, 89 ; M_{43}, 10.
,, (2)	16·2	$4{\cdot}0\times10^{24}$	43·85	2·40	M_{44}, 84 ; M_{43}, 15.
,, (3)	40·6	$1{\cdot}0\times10^{25}$	43·79	2·40	M_{44}, 79 ; M_{43}, 19.
,, (4)	65	$1{\cdot}6\times10^{25}$	43·78	2·40	M_{44}, 78·5 ; M_{43}, 19·5.
,, (5)	162	$4{\cdot}0\times10^{25}$	43·84	2·40	M_{44}, 82 ; M_{43}, 15.
,, (6)	403	$1{\cdot}0\times10^{26}$	44·08	2·40	M_{45}, 12 ; M_{44}, 83 ; M_{43}, 5.

Such densities at this temperature will be found as mean conditions in stars of approximately the following values of $M^*/\odot$:—

(1) 2·3, (2) 1·5, (3) 0·8, (4) 0·6, (5) 0·35, (6) 0·25.

(These values are only roughly interpolated from fig. 1.)

Table VII. For Fe at $5{\cdot}64\times10^6$ °K the density can be altered in the ratio 10^4 to 1 with only a 2 per cent. change in mean molecular weight. This is rather a favourable case. We have taken some care to secure for comparison a particularly *unfavourable* case and have chosen Br at 4×10^6 °K. At this temperature between the densities 0·0004 and 0·5 Br is changing from mainly Li-like to mainly Na-like, and μ increases by 25 per cent. Even here the density effect is not great and it can scarcely ever be greater for any element. As a direct consequence of this insensitivity we deduce the important approximation that *for a given temperature and density the mean degree of ionisation of any element*

* *Loc. cit.*

TABLE VII.

Change of μ for Change of Density at Constant Temperature.

Fe. $T = 5{\cdot}64 \times 10^6$ °K.

ρ. gm./c.c.	Free Electrons per c.c.	s.	μ.
4·06	$1{\cdot}00 \times 10^{24}$	22·64	2·36
·0678	$1{\cdot}69 \times 10^{22}$	22·97	2·33
·0004	$1{\cdot}00 \times 10^{20}$	23·09	2·32

Br. $T = 4{\cdot}00 \times 10^6$ °K.

ρ. gm./c.c.	Free Electrons per c.c.	s.	μ.	Chief Types of Ions in Percentages.
0·00042	$1{\cdot}00 \times 10^{20}$	31·48	2·46	M_{32}, 49 ; M_{31}, 44.
0·0047	$1{\cdot}04 \times 10^{21}$	29·27	2·74	M_{31}, 21 ; M_{30}, 19 ; M_{29}, 28 ; M_{28}, 19 ; M_{27}, 9.
0·0506	$1{\cdot}00 \times 10^{22}$	26·07	2·95	M_{28}, 7 ; M_{27}, 24 ; M_{26}, 36 ; M_{25}, 29.
0·528	$1{\cdot}00 \times 10^{23}$	24·98	3·08	M_{26}, 17 ; M_{25}, 56 ; M_{24}, 23.

will be practically independent of the presence of other elements except perhaps in its transition regions. The accuracy of this approximation is guaranteed by the fact that at a given temperature and density N/V never varies in our tables by a factor so large as 1·3 for assumed compositions of pure Fe, O, and Ag, whereas it would not matter if it varied by a factor of 100 in most cases, or at any rate in special regions 2 or 3.* For this reason it is not necessary to carry out detailed calculations for mixtures, for the mean molecular weight can be calculated for any assumed composition by simple proportional parts.

From the results for pure elements, even for compositions containing a very high percentage of hydrogen, the mean degree of dissociation for the other elements present will in general not be appreciably altered, though of course the value of the mean molecular weight will be affected. This is clearly shown by the examples of mixtures in Table I. These are the only examples for mixtures that we have thought it worth while working out in detail ; it will be noticed that the maximum alteration in the mean degree of dissociation of one element due to the presence of another element is about 0·1 per cent. It would be greater in regions of high density where the electrostatic correction becomes important, but even there it would still be unimportant.

We consider that our results are sufficient to justify interpolation and to some extent extrapolation within the range of temperatures and densities ordinarily occurring in giant and dwarf stars, and that the errors thereby involved will in all cases be negligible compared with those due to uncertainty of the composition of the stars. For convenience in interpolation and extrapolation we give charts showing how the mean molecular weight depends on the temperature and density, both the latter being plotted on a logarithmic scale. The points for which the mean molecular weight has been calculated are marked by dots and the values obtained are shown against them. In order to

* A change of material from Fe to H for given density will only increase N by a factor just greater than 2.

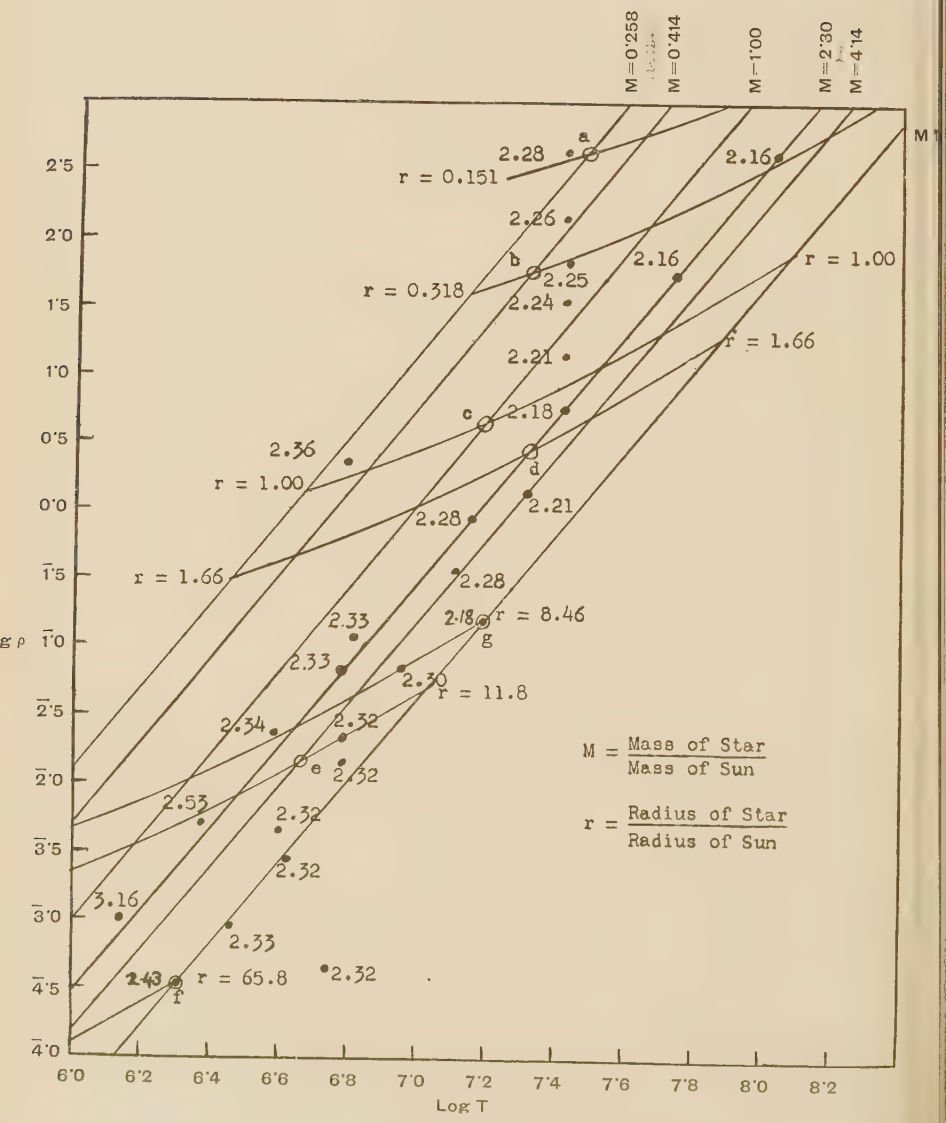

FIG. 1.—Stars of Fe. The plotted points show the calculated values of the mean molecular weight for given T and ρ. The grid shows the mean values of T and ρ for stars of given mass and given radius. An approximate weight for a star of given mass and radius can be directly interpolated from the grid and the plotted points.

show that the range of our calculations covers the conditions usually met with in stars, we have also drawn curves corresponding to average conditions for stars of constant mass (these are straight lines of gradient 3) and of constant radius respectively. This grid is based on the table vii. of Eddington's last paper, and is only intended to be approximate. It is based on a mean molecular weight 2·11 and the mean densities and temperatures assigned by Eddington for seven typical stars. These stars are also shown in the figures by the letters (*a*), (*b*), . . . (*g*). They are—

(*a*) Krueger 60.
(*b*) η Cassiop., *f*.
(*c*) The Sun.
(*d*) Sirius.
(*e*) Capella.
(*f*) Typical Cepheid.
(*g*) Typical Eclipsing Variable.

All points are well covered by the calculations. Further calculations might perhaps be desirable for exceptionally massive stars, and of course for white dwarfs. Otherwise the ordinary range of stars is catered for. The grid, though only approximate, could be safely used as a means of selecting a value of μ with which to begin calculations for a star of assigned mass and radius.

§ 4. *Deviations from the Perfect Gas Laws.*—It remains to consider deviations of the gas-pressure from that of a perfect gas. We use the equation

$$P_G V=(P-P_R)V=kT\{(N+M)+a-b\} \quad . \quad . \quad (14)$$

in which P is the total pressure, P_G and P_R the gas- and radiation-pressures, a the correction term due to excluded volumes, and $(-b)$ the correction term due to the electrostatic charges. The first term on the right of (14) would give the pressure if the matter could be treated as a dissociating but otherwise perfect gas. Different possible theories give different values for a. The theory we have adopted here in fixing $u_r(T)$ gives for the main contribution to a the simple expression $\frac{2}{3}M$, which is at the same time an upper limit to a and an approximation (on this theory) generally close to the true value. It gives the correct total effect in any case in which all the important $u_r(T)$ are large compared to $(\varpi_r)_0$. For any stage for which the dominant term in $u_r(T)$ is $(\varpi_r)_0$, the contribution to a is zero. In our actual calculations almost the only important stages of this type are the bare O-nuclei, but we have not made any correction for these in the tables of this section. An additional contribution to a arises from the volumes of the atomic cores. This is strictly analogous to the effect of the ordinary atomic volumes of van der Waals' theory, and is considered separately later on.

The second correction term $(-b)$ is given by

$$b=\frac{\sqrt{\pi}\epsilon^3}{3V^{\frac{1}{2}}(kT)^{\frac{3}{2}}}\left\{N+\Sigma_r r^2 M_r\right\},$$

or with sufficient accuracy,

$$b=6{\cdot}428\times10^{-29}\frac{N^{\frac{3}{2}}(1+s)^{\frac{5}{2}}}{V^{\frac{1}{2}}(kT)^{\frac{3}{2}}}.$$

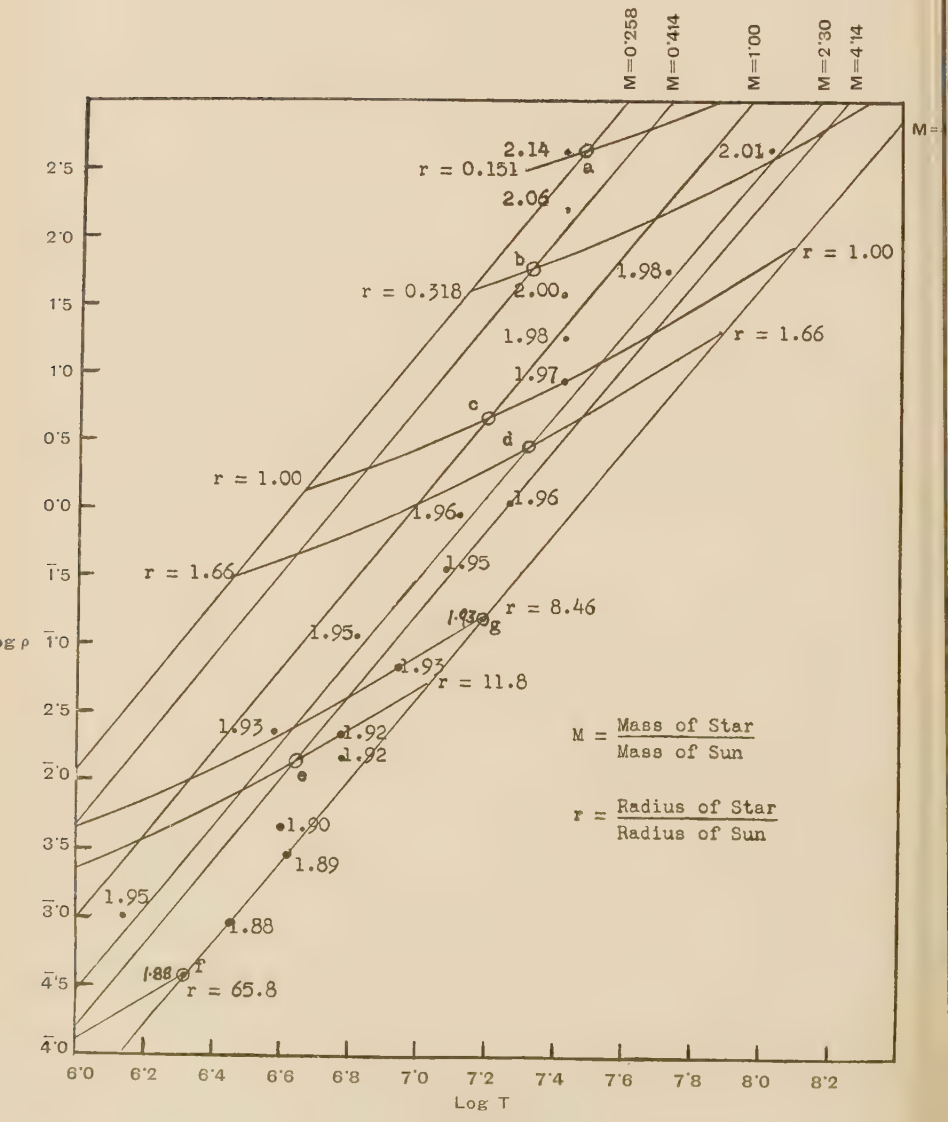

FIG. 2.—Stars of O. [*See* legend of fig. 1.]

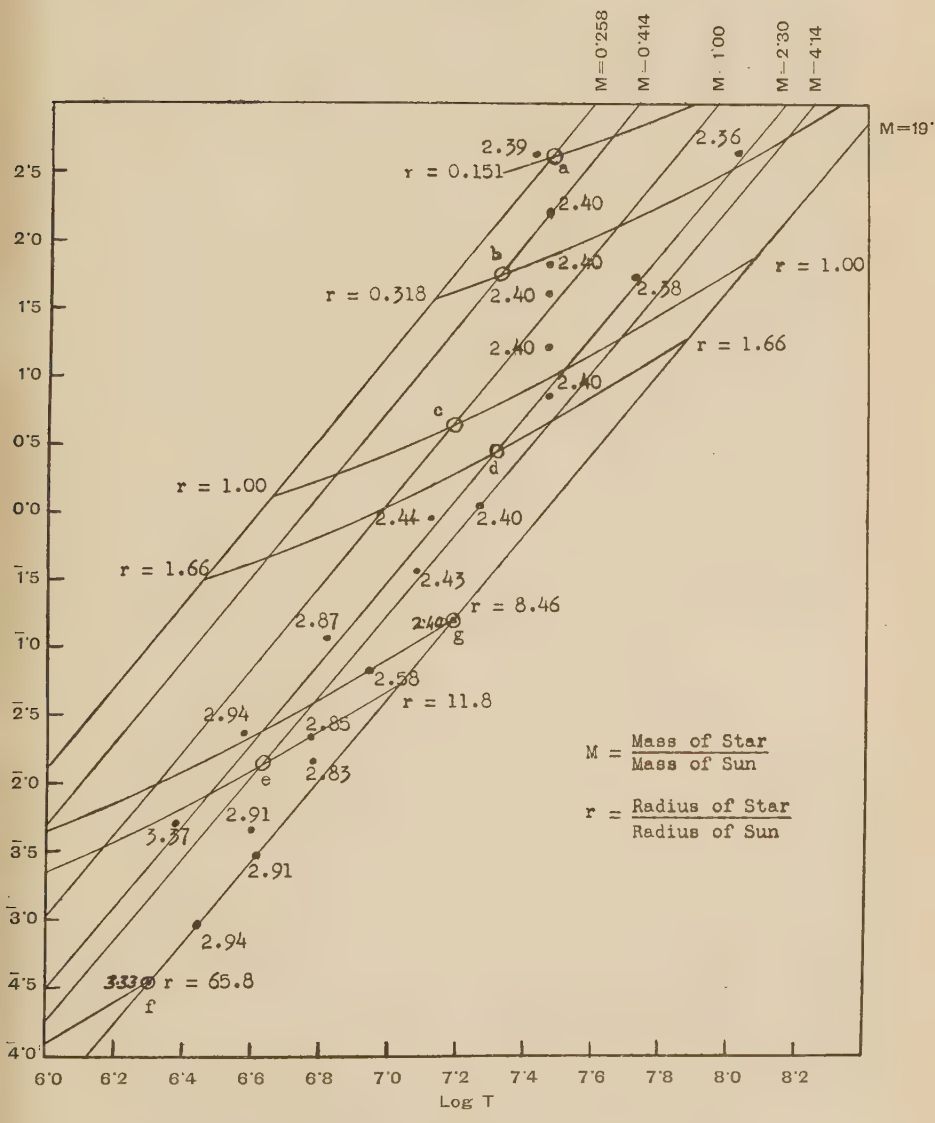

FIG. 3.—Stars of Ag. [*See* legend of fig. 1.]

The results of calculations of pressure for the conditions of Table II. are given in Table VIII.

TABLE VIII.

Contributions to the Pressure, under the Conditions of Table II.

Temperature [Densities].	Composition.	N+M per c.c.	$\frac{2}{3}$M per c.c. (*a*).	Electrostatic Term per c.c. (*b*).	% Corrections. (*a*).	(*b*).	(*a*)−(*b*).
$6{\cdot}59\times10^{6}$	Fe	$2{\cdot}82\times10^{22}$	$7{\cdot}8\times10^{20}$	$1{\cdot}23\times10^{21}$	2·8	4·4	− 1·6
[0·1085]	O	$3{\cdot}38\times10^{22}$	$2{\cdot}74\times10^{21}$	$2{\cdot}86\times10^{20}$	8·1	0·8	+ 7·3
	Ag	$2{\cdot}29\times10^{22}$	$4{\cdot}06\times10^{20}$	$1{\cdot}56\times10^{21}$	1·8	6·8	− 5·0
$1{\cdot}318\times10^{7}$	Fe	$2{\cdot}29\times10^{23}$	$6{\cdot}26\times10^{21}$	$1{\cdot}04\times10^{22}$	2·7	4·5	− 1·8
[0·868]	O	$2{\cdot}69\times10^{23}$	$2{\cdot}19\times10^{22}$	$2{\cdot}25\times10^{21}$	8·1	0·8	+ 7·3
	Ag	$2{\cdot}16\times10^{23}$	$3{\cdot}25\times10^{21}$	$2{\cdot}38\times10^{22}$	1·5	11·0	− 9·5
$2{\cdot}636\times10^{7}$	Fe	$1{\cdot}93\times10^{24}$	$5{\cdot}02\times10^{22}$	$9{\cdot}65\times10^{22}$	2·6	5·0	− 2·4
[6·94]	O	$2{\cdot}13\times10^{24}$	$1{\cdot}75\times10^{23}$	$1{\cdot}75\times10^{22}$	8·2	0·8	+ 7·4
	Ag	$1{\cdot}75\times10^{24}$	$2{\cdot}60\times10^{22}$	$1{\cdot}99\times10^{23}$	1·5	11·4	− 9·9
$5{\cdot}27\times10^{7}$	Fe	$1{\cdot}55\times10^{25}$	$4{\cdot}00\times10^{23}$	$7{\cdot}95\times10^{23}$	2·6	5·1	− 2·5
[55·5]	O	$1{\cdot}70\times10^{25}$	$1{\cdot}41\times10^{24}$	$1{\cdot}36\times10^{23}$	8·3	0·8	+ 7·5
	Ag	$1{\cdot}40\times10^{25}$	$2{\cdot}06\times10^{23}$	$1{\cdot}62\times10^{24}$	1·5	11·6	−10·1
$1{\cdot}054\times10^{8}$	Fe	$1{\cdot}24\times10^{26}$	$3{\cdot}21\times10^{24}$	$6{\cdot}32\times10^{24}$	2·6	5·1	− 2·5
[444]	O	$1{\cdot}42\times10^{26}$	$1{\cdot}12\times10^{25}$	$1{\cdot}22\times10^{24}$	7·9	0·9	+ 7·0
	Ag	$1{\cdot}14\times10^{26}$	$1{\cdot}66\times10^{24}$	$1{\cdot}35\times10^{25}$	1·5	11·8	−10·3

The two corrections are, of course, of opposite sign and both are very nearly a constant fraction of the pressure over a wide range of temperatures and densities on this sequence. If there were no changes of s then, of course, the corrections would necessarily be absolutely constant along the sequence which relates to homologous stars in which T is varied by a factor α and ρ by a factor α^3. Such factors also leave the ratio of the electrostatic term to N+M unaltered *except for changes in s*. Even the sudden change in the entry for Ag between the temperatures $6{\cdot}59\times10^{6}$ and $1{\cdot}318\times10^{7}$ is simply due to this change of s, for between these temperatures on this sequence the average state of Ag changes from Na-like to Li-like. It will be observed that for a star of a slightly lighter element than Fe (somewhere about Sc) the two corrections will balance, and the gas will behave almost exactly as if it were perfect (at its Boyle point). The orders of these corrections will hardly change appreciably if we pursue this sequence to much higher temperatures and densities.

If only the excluded volume correction had to be made, as is the case for stars of very low temperatures or very light elements, this correction could be made with complete accuracy by "faking" the mean molecular weight. For the actual gas-pressure is then that of a perfect gas with N + $\frac{2}{3}$M particles in a volume V instead of N + M, and

therefore it behaves as if its mean molecular weight were $A/(\frac{5}{3}+s)$ instead of $A/(1+s)$. In general, however, we must expect the electrostatic correction to be of somewhat the greater importance, and both corrections must be retained. The full correction, too, under the conditions here investigated, can probably be best allowed for by "faking" the mean molecular weight; the "fake" is now, of course, merely a convenient empirical approximation, justified by the noteworthy constancy of the combined correction. The effective mean molecular weight obtained thus is

$$\frac{100\mu}{100\pm x},$$

where μ is the true mean molecular weight, and x is the entry in the last column of Table VIII.

Along this sequence of conditions (centres of stars of mass $1{\cdot}5\odot$) we have worked up to densities of 440 (and could go almost indefinitely further) without encountering any serious deviations from the perfect gas laws for stars of normal constitution (average material Fe). More severe densities will occur at lower temperatures for stars of smaller mass. Before passing on to these, however, it is desirable to examine possible contributions to the excluded volumes due to the atomic cores that remain intact. These are additional to the formal excluded volumes hitherto dealt with which arise as a contribution from the presence of an electron in an orbit of high excitation in the majority of atoms present. These core volumes are, as we have said, the strict analogues of the ordinary atomic volumes familiar to us at ordinary temperatures. As an example, consider Fe at the density 444 in Table II., where it appears that most of the Fe is H-like, and some 20 per cent. He-like. If we assume that all the Fe is He-like and all like He in its normal state, this will certainly overestimate the average volume of the remaining atomic cores. On this assumption we can estimate the extra effect of these cores on the pressure with some pretence at accuracy.

The general formula for the pressure of a gas mixture of which the constituent systems have rigid volumes is

$$\frac{p\mathrm{V}}{k\mathrm{T}}=\Sigma_\alpha \mathrm{N}_\alpha+\frac{1}{\mathrm{V}}\Sigma_{\alpha\beta}\frac{\mathrm{N}_\alpha \mathrm{N}_\beta}{\sigma_{\alpha\beta}}v_{\alpha\beta} \quad . \quad . \quad . \quad (15)$$

correct to terms in 1/V. In (15) N_α is the number of molecules of type α in a volume V, $v_{\alpha\beta}$ is the volume denied to the centre of an α (or β) by the presence of a β (or α) and $\sigma_{\alpha\alpha}=2$, $\sigma_{\alpha\beta}=1(\alpha\neq\beta)$. The summation $\Sigma_{\alpha\beta}$ is extended over all pairs of types. For a gas of M He-like Fe ions plus N electrons ($\mathrm{N}=24\mathrm{M}$),

$$\frac{p\mathrm{V}}{k\mathrm{T}}=\mathrm{N}+\mathrm{M}+\frac{1}{\mathrm{V}}\{\tfrac{1}{2}\mathrm{N}^2(0)+\mathrm{NM}v_{ec}+\tfrac{1}{2}\mathrm{M}^2v_{cc}\}. \quad . \quad . \quad (15.1)$$

where v_{ec} is the volume denied by the Fe-core to an electron, and v_{cc} that denied to another Fe-core. It is perhaps legitimate to suppose that $v_{ec}=\frac{1}{8}v_{cc}$. The "volume" of the electron itself may be taken to

be zero. The ratio of the correcting term to the main term in (15.1) is therefore effectively Mv_{cc}/V. In the same way the ratio of the correcting term to the main term for normal He is $\frac{1}{2}M'v'_{cc}/V$. Now the value of b in van der Waals' equation for 1 c.c. of normal He at normal pressure and temperature is about $4{\cdot}1\times10^{-4}$, corresponding to a density of 0·000178 gm./c.c.* The effect of the core volumes therefore amounts to a 1 per cent. correction at a density of $0{\cdot}000178\times\frac{100}{4{\cdot}1}$ for He, and at a density of $0{\cdot}000178\times\frac{100}{4{\cdot}1}\times\frac{56}{4}$ for He-atoms of the mass of Fe. This would be the required density here if $v_{ee}(=\frac{1}{8}v_{cc})$ and $\frac{1}{2}v'_{cc}$ were the same. But He-like Fe is smaller than He. It is almost certain on general evidence that the radius of the atom in this connection is closely proportional to the radius of the electronic orbits, and this radius is inversely proportional to $Z-\gamma$, where Z is the nuclear charge and γ a screening constant. We can estimate γ from the negative energy of He itself, for we have $24{\cdot}5+54{\cdot}2=2\times13{\cdot}54(2-\gamma)^2$, $\gamma=0{\cdot}3$. The radius of the orbits of He-like Fe is, therefore, smaller in the ratio $(26-\gamma)/(2-\gamma)=15{\cdot}1$. Hence the 1 per cent. correction will only be reached in the He-like Fe (plus free electrons) at a density of

$$0{\cdot}000178\times\frac{100}{4{\cdot}1}\times\frac{56}{4}\times(15{\cdot}1)^3\times4=840.$$

In this rough calculation we have ignored the electrostatic forces which, apart from their direct effect on the pressure already calculated, will modify (by increasing) the effect of the excluded volumes of the cores. It is difficult to foresee the exact size of this correction, which we shall not investigate further here.

The correction as calculated would, therefore, reach $\frac{1}{2}$ per cent. in this star, but we have taken a wide overestimate. In fact one can see that it is not likely to reach 2 per cent. along this sequence until the density reaches perhaps 5000.† At higher densities the core volumes will be more important, and at densities of 50,000 in a star of mass $1{\cdot}5\odot$ we might expect to find that serious diminutions in compressibility have begun to set in, with the effect of the remaining core volumes overweighting the electrostatic correction to the pressure.

The core-volume effect as here crudely estimated would be smaller for the silver star and larger for the oxygen star. It does not seem worth while to pursue the matter further without a more adequate theoretical discussion; it is clear that, as Eddington has pointed out, these high densities are quite possible in gaseous stars, and that the gas is not so far removed from "perfection." It is, of course, the core-

* Kaye and Laby (1918), p. 10.

† The H-like Fe will in general be highly excited and have no core at all. The remaining $\frac{1}{3}$th of He-like Fe will also be highly excited and perhaps have an H-like core which might be expected to possess an effective volume even less than the He-like core. Thus there is a factor of safety of about 2 in the estimate made in the text.

volume effect which is really important. The excluded volume term of the theory we have used here, due to the excited states, is really trivial, as its importance does not rise with the density. An alteration of the theory which alters or abolishes this term will be without effect in these applications.

The results for higher densities for less massive stars are shown in Table IX ; the data correspond to those of Table VI. ; the temperature is the same throughout, and is approximately that for the sun.

TABLE IX.

$T = 2{\cdot}636 \times 10^7$.

Element.	ρ.	N+M per c.c.	$\frac{3}{2}$ M per c.c. (*a*).	Electrostatic Terms per c.c. (*b*).	% Corrections. (*a*).	(*b*).	(*a*)−(*b*).
(1) Fe	6·95	$1{\cdot}93\times10^{24}$	$5{\cdot}02\times10^{22}$	$9{\cdot}60\times10^{22}$	2·6	5·0	− 2·4
(2) Fe	15·2	$4{\cdot}16\times10^{24}$	$1{\cdot}10\times10^{23}$	$3{\cdot}00\times10^{23}$	2·6	7·2	− 4·6
(3) Fe	38·4	$1{\cdot}04\times10^{25}$	$2{\cdot}78\times10^{23}$	$1{\cdot}17\times10^{24}$	2·7	11·2	− 8·5
(4) Fe	71·6	$1{\cdot}93\times10^{25}$	$5{\cdot}40\times10^{23}$	$2{\cdot}89\times10^{24}$	2·8	14·9	−12·1
(5) Fe	156	$4{\cdot}16\times10^{25}$	$1{\cdot}17\times10^{24}$	$9{\cdot}12\times10^{24}$	2·8	21·9	−19·1
(6) Fe	392	$1{\cdot}04\times10^{26}$	$2{\cdot}95\times10^{24}$	$3{\cdot}58\times10^{25}$	2·8	34·4	−31·6
(1) O	6·97	$2{\cdot}13\times10^{24}$	$1{\cdot}75\times10^{23}$	$1{\cdot}75\times10^{22}$	8·2	0·8	+ 7·4
(2) O	18·6	$5{\cdot}71\times10^{24}$	$4{\cdot}70\times10^{23}$	$7{\cdot}57\times10^{22}$	8·2	1·3	+ 6·9
(3) O	37·6	$1{\cdot}14\times10^{25}$	$9{\cdot}45\times10^{23}$	$2{\cdot}12\times10^{23}$	8·3	1·9	+ 6·4
(5) O	156	$4{\cdot}59\times10^{25}$	$3{\cdot}96\times10^{24}$	$1{\cdot}61\times10^{24}$	8·6	3·5	+ 5·1
(6) O	408	$1{\cdot}15\times10^{26}$	$1{\cdot}03\times10^{25}$	$6{\cdot}04\times10^{24}$	9·0	5·2	+ 3·8
(1) Ag	6·94	$1{\cdot}75\times10^{24}$	$2{\cdot}60\times10^{22}$	$1{\cdot}98\times10^{23}$	1·5	11·4	− 9·9
(2) Ag	16·2	$4{\cdot}09\times10^{24}$	$6{\cdot}07\times10^{22}$	$7{\cdot}10\times10^{23}$	1·5	17·4	−15·9
(3) Ag	40·6	$1{\cdot}02\times10^{25}$	$1{\cdot}52\times10^{23}$	$2{\cdot}80\times10^{24}$	1·5	27·4	−25·9
(4) Ag	65	$1{\cdot}64\times10^{25}$	$2{\cdot}43\times10^{23}$	$5{\cdot}65\times10^{24}$	1·5	34·4	−32·9
(5) Ag	162	$4{\cdot}09\times10^{25}$	$6{\cdot}07\times10^{23}$	$2{\cdot}25\times10^{25}$	1·5	55·0	−53·5
(6) Ag	403	$1{\cdot}02\times10^{26}$	$1{\cdot}51\times10^{24}$	$8{\cdot}94\times10^{25}$	1·5	87·5	−86·0

Such densities at this temperature will be found as mean conditions in stars of approximately the following values of $M^*/\odot$:—

(1) 2·3, (2) 1·5, (3) 0·8, (4) 0·6, (5) 0·35, (6) 0·25.

(These values are only roughly interpolated from fig. 1.)

We see that at this temperature there is no very great deviation from the perfect gas laws up to a density of 60, which would be the mean density for η Cassiop. (*f*), except for elements of high atomic number such as silver. If, however, the star can be fairly represented as being composed of elements in the neighbourhood of Fe, or if there are other elements present in appreciable quantity of very high and very low atomic number, provided the two classes are present in comparable amounts, then at this temperature and a density of 60 the deviations from the perfect gas laws are almost certainly still small.

At still greater densities of about 400, as in Krueger 60, there are, as here calculated, large deviations from the perfect gas laws in the perhaps unexpected direction of greater than normal compressibility. It must be remembered, however, that there is still a correction to be applied for the core volumes, which can only be very crudely estimated in the face of so large an electrostatic correction. It must be remembered too that we have here reached a region in which the theory, on which the calculation of the electrostatic term itself is based, has been pushed beyond its limits of reliability.* If in the face of these uncertainties an estimate is still required, we find from Table VI. for the Fe-star that we may still assume that the cores are all He-like. The core-volume correction is, therefore, still only $\frac{1}{2}$ per cent., but here it seems likely that it is effectively of this order and not much smaller than it was before. In such a star as Krueger 60 therefore ($M^*/\odot$ -0·25) the principal deviation from the perfect gas law is practically the pure electrostatic correction. For still denser stars of this mass the theory suggests a falling off in compressibility at perhaps a density of 15,000–20,000. This is very vague ; one can perhaps safely say only that the theory suggests that the compressibility begins to fall off at lower densities for stars of smaller mass.

Summary.—The paper presents the results of an extensive series of calculations of the mean molecular weight of matter of various constitutions at the temperatures and densities found in gaseous stars. They are shown in tabular and diagrammatic form.

The tables show also the distribution of the various atoms among their principal stages of ionisation.

The deviations of the gas from "perfection" are also discussed. It is shown that the most important deviation is due to the electrostatic forces which form polarisation fields round each ion. These give rise to a correcting term diminishing the pressure, whose ratio to the pressure practically depends only on the mass of the star, and is greatest for stars of small mass, becoming important for stars less massive than the Sun.

It is estimated that the portions of the electronic structures of the atoms which remain *in situ* will have begun to produce noticeable diminutions in compressibility by densities of 50,000 in stars of mass $1\frac{1}{2}$ times the Sun's mass and rather sooner in less massive stars. At densities of 5000 in such stars this effect cannot have begun.

* The condition is roughly that the correction to the gas laws should be small. See Fowler, *Proc. Camb. Phil. Soc.* (1925), in publication.

Applications of Statistical Mechanics to Determine the Properties of Matter in Stellar Interiors. Part II. *The Adiabatics.* By R. H. Fowler and E. A. Guggenheim.

§ 1. *Introduction.*—In the preceding paper * we have studied the dissociative equilibrium and equation of state of gaseous matter at stellar temperatures, on the basis of the theories of statistical mechanics,† for application to Eddington's theory of the radiative equilibrium of a star. In the present paper we extend the discussion to determine the form of the (P, V)-adiabatic curves of the stellar material, for application to Eddington's theory of pulsating stars ‡ (Cepheid variables).

It is hardly necessary to present an elaborate introduction to this problem. It will be sufficient to record the following formulae. In Eddington's work on the Cepheids the equation of the (P, V)-adiabatics is written

$$\frac{dP}{P}=\gamma\frac{d\rho}{\rho}=-\gamma\frac{dV}{V} \quad . \quad . \quad . \quad . \quad (1)$$

where P is the total pressure, including the pressure of radiation, and ρ and V are the density and volume of the material under consideration. Equation (1) defines γ. It is a well-known thermodynamic result that

$$\gamma=-\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_S=\frac{C_P}{C_V}\left\{-\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T\right\} \quad . \quad . \quad . \quad (2)$$

In equation (2) C_P and C_V are the ordinary specific heats at constant total pressure and volume, and S and T denote the entropy and temperature.

The specific heats themselves may be taken as defined by the equations

$$C_V=\left(\frac{\partial E}{\partial T}\right)_V, \qquad C_P=\left(\frac{\partial I}{\partial T}\right)_P \quad . \quad . \quad . \quad (3)$$

where E is the internal energy, and I the "total heat." It will be noticed that, unless the isothermals, as for a perfect non-dissociating gas, have the form PV=const., γ is not equal to the ratio of the specific heats. To determine γ we have to calculate the three factors in (2) from the equations (3) and the equation of state, which can be done for any not too dense stellar material for which the degree of ionisation has been determined in the preceding paper.

If the assembly considered contains radiation but no matter, equation (2) becomes illusory, since $(\partial P/\partial V)_T=0$ and C_P is infinite. It is well known that the γ defined by (1) for pure radiation has the value $\frac{4}{3}$, and it is easy to check by direct calculation that the value of γ given by (2) for, say, radiation, and a perfect monatomic gas of

* *M.N.*, **85**, 939.
† Fowler, *Proc. Camb. Phil. Soc.* (1925); and *Phil. Mag.* (in publication).
‡ Eddington, *M.N.*, **79**, 2, 177 (1919).

M atoms tends to the value $\frac{4}{3}$ as $M \longrightarrow 0$. The indeterminacy of (2) for pure radiation is, therefore, merely superficial.

The exact calculation of γ for ionised gases, even very dilute ones, would be a complicated affair. The calculations can, however, be very greatly simplified for the present purpose by the following considerations. We have found that over the important part of a range * of temperatures from 10^6 °K to 10^8 °K, and of densities from 0·005 to 400, the degree of ionisation changes only very slowly. For example, the average number of electrons lost by iron atoms over this range remains between 21 and 25. Moreover, if the average number of electrons lost by any atom lies between the integers r and $r+1$, then in general more than 90 per cent. of these atoms have lost either r or $r+1$ electrons. We shall, therefore, assume as an approximation that every atom of this element has lost either r or $r+1$ electrons. Further still it can be shown that even if in reality there is an appreciable number of atoms in a third state, this will only occur if the energy changes involved in the two transitions between the two pairs of consecutive states are roughly equal. In this case the error in assuming only two states in calculating γ must be small and becomes negligible if the formulæ are applied judiciously ; this will be made clearer later.

We have also shown in the previous paper that if the gaseous pressure be calculated as for a partially dissociated gas, in which every constituent is " perfect," the error is never serious in the range of temperatures and densities occurring in the giant stars. We shall, therefore, ignore this error also, and treat our matter as a dissociating but otherwise perfect gas, in which the constituents are electrons and atoms in just two consecutive stages of ionisation, returning in a later section to discuss cases with more than two stages effectively present.

§ 2. *Specification of a Simplified Assembly.*—We consider then an assembly of radiation and M atoms of which M_1 have lost r and M_2 have lost $r+1$ electrons ($M_1+M_2=M$). We denote M_2/M_1 by x. The number N of free electrons is given by

$$N=rM_1+(r+1)M_2=rM\left\{1+\frac{1}{r}\frac{x}{1+x}\right\} \quad . \quad . \quad (4)$$

The equation of dissociative equilibrium is

$$\frac{N}{V}\frac{M_2}{M_1}=K(T)=\frac{(2\pi mkT)^{\frac{3}{2}}}{h^3}\frac{u_{r+1}(T)}{u_r(T)}\exp.\left\{-\frac{\chi_r}{kT}+L\right\} \quad . \quad (5)$$

The various expressions have their usual meanings.† In particular χ_r is the work required to remove the $r+1^{th}$ electron from the r-times ionised atom ; L is the correction term for the electrostatic charges on the ions and electrons. Over the range of temperatures and densities occurring in Cepheids L is unimportant compared with χ_r/kT, and we shall neglect it ; we have verified that the error introduced is at most 2 per cent. for the highest densities.

* See charts in our previous paper.

† Fowler and Guggenheim, *loc. cit.*

With regard to $u_r(T)$ we have seen * that in general we can assume, with sufficient accuracy, either that $u_r(T)$ is constant—most of the r-times ionised atoms in their normal unexcited state—or that

$$u_r(T) \propto T\left(\frac{V}{N}\right)^{\frac{3}{2}} e^{-\chi_r/kT} \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) holds when T is so large that most of the r-times ionised atoms are in excited states. The function $u_{r+1}(T)$ is of the same form. There are, therefore, three cases to be considered :

A. Both $u_r(T)$ and $u_{r+1}(T)$ are constant.
B. Both $u_r(T)$ and $u_{r+1}(T)$ satisfy an equation of the form (6).
C. The function $u_r(T)$ satisfies (6) and $u_{r+1}(T)$ is constant. The converse form of *C* cannot occur.

We shall at each stage aim at a maximum permissible error of 2 per cent., most of the errors lying far below this limit. We attain this degree of accuracy if we always neglect terms in r^{-3} and further ignore all terms in r^{-2}, except such as are multiplied by a factor of the type $\{1{\cdot}5+\chi/kT\}^2$. The maximum error introduced by this is of the order 1 per cent. for oxygen, but for iron and elements of higher atomic number it is only of the order 0·1 per cent.

We shall follow Eddington in using β for the ratio of the gas-pressure, P_G, to the total pressure. Thus

$$\beta = P_G/P, \quad P = P_G + P_R, \quad . \quad . \quad . \quad (7)$$

where P_R is the radiation pressure. Then

$$P_G = (N+M)kT/V, \quad P_R = \tfrac{1}{3}aT^4, \quad . \quad . \quad (8)$$

where a is Stefan's constant.

§ 3. *Calculation of the Isothermal Factor.*—From equations (7) and (8)

$$-\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T = -\beta\frac{V}{P_G}\left(\frac{\partial P_G}{\partial V}\right)_T = \beta\left\{1 - \frac{V}{N+M}\left(\frac{\partial N}{\partial V}\right)_T\right\} \quad . \quad (9)$$

From equation (4)

$$dN = \frac{M\,dx}{(1+x)^2} \quad . \quad . \quad . \quad . \quad (10)$$

Thus the second term in { } in (9) is of order $1/r$, and we may write

$$-\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T = \beta\left\{1 - \frac{1}{r}\frac{V}{(1+x)^2}\left(\frac{\partial x}{\partial V}\right)_T\right\} \quad . \quad . \quad (11)$$

In calculating $(\partial x/\partial V)_T$ from (4) and (5) terms of order $1/r$ may themselves be neglected, so that we have

$$x \propto VK(T).$$

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* Fowler, *loc. cit.*

In cases A and B, K(T) is independent of V, and so $x \propto V$. In case C, on account of (6), $x \propto V^{\frac{1}{3}}$. We therefore have

$$-\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T=\beta\left\{1-\frac{1}{r}\frac{x}{(1+x)^2}\right\} \quad . \quad (12)\ (A, B)$$

$$-\frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T=\beta\left\{1-\frac{1}{3r}\frac{x}{(1+x)^2}\right\} \quad . \quad (12)\ (C)$$

Equations (12) enable us to evaluate this factor in γ numerically. It is never seriously different from β; in fact never more than 2 per cent.

§ 4. *Calculations of* C_V.—The general formulæ for the calculation of the energy content of the whole assembly as a function of the temperature * give us

$$E=\tfrac{3}{2}(N+M)kT-M_1(\chi_r+\chi_{r+1}+\ \ldots\ +\chi_{z-1})+M_1kT^2\frac{\partial}{\partial T}\log u_r(T)-$$

$$-M_2(\chi_{r+1}+\chi_{r+2}\ \ldots\ +\chi_{z-1})+M_2kT^2\frac{\partial}{\partial T}\log u_{r+1}(T)+aVT^4 \quad (13)$$

The partial differentiation of $u_r(T)$ is carried out so that only the factors $e^{\{\chi_r-(\chi_r)_s\}/kT}$ in the various terms are to be differentiated.

In case A u_r and u_{r+1} reduce to their first (constant) terms, so that equation (13) becomes

$$E=\tfrac{3}{2}(N+M)kT-M(\chi_{r+1}+\chi_{r+2}+\ \ldots\ +\chi_{z-1})-M_1\chi_r+aVT^4\ ;$$

since M is constant and $dM_1=-dN$, we obtain

$$C_V=\tfrac{3}{2}(N+M)k+4aVT^3+(\tfrac{3}{2}kT+\chi_r)\left(\frac{\partial N}{\partial T}\right)_V \quad . \quad . \quad (14)\ (A)$$

In case B we have large numbers of atoms in both stages in highly excited states, so that

$$kT^2\frac{\partial}{\partial T}\log u_r(T)=\chi_r,\quad kT^2\frac{\partial}{\partial T}\log u_{r+1}(T)=\chi_{r+1},$$

$$E=\tfrac{3}{2}(N+M)kT-M(\chi_{r+2}+\ \ldots\ +\chi_{z-1})-M_1\chi_{r+1}+aVT^4,$$

$$C_V=\tfrac{3}{2}(N+M)k+4aVT^3+(\tfrac{3}{2}kT+\chi_{r+1})\left(\frac{\partial N}{\partial T}\right)_V \quad . \quad . \quad (14)\ (B)$$

To calculate $(\partial N/\partial T)_V$ we use (4), (5), and (10), retaining terms in $1/r$. The relation between x and T is now (case A)

$$x\left(1+\frac{1}{r}\frac{x}{1+x}\right)\propto T^{\frac{3}{2}}e^{-\chi_r/kT}, \quad . \quad . \quad . \quad (15)$$

and we find

$$\left(\frac{\partial N}{\partial T}\right)_V=\frac{M}{T}\frac{x}{(1+x)^2}\left\{\frac{3}{2}+\frac{\chi_r}{kT}\right\}\left\{1-\frac{1}{r}\frac{x}{(1+x)^2}\right\} \quad . \quad (16)$$

$$\frac{C_V}{(N+M)k}=\frac{3}{2}+12\left(\frac{1}{\beta}-1\right)+\frac{M}{N+M}\left(\frac{3}{2}+\frac{\chi_r}{kT}\right)^2\frac{x}{(1+x)^2}\left\{1-\frac{1}{r}\frac{x}{(1+x)^2}\right\} \quad (17)\ (A, B)$$

Case B differs only in χ_r being replaced by χ_{r+1}.

* Fowler, *loc. cit.*

In case *C*, $kT^2 \frac{\partial}{\partial T} \log u_r(T) = \chi_r$ and $u_{r+1}(T)$ is a constant, so that

$$E = \tfrac{3}{2}(N+M)kT - M(\chi_{r+1}+\chi_{r+2}+ \ldots +\chi_{s-1}) + aVT^4,$$

$$C_V = \tfrac{3}{2}(N+M)k + 4aVT^3 + \tfrac{3}{2}kT\left(\frac{\partial N}{\partial T}\right)_V \quad . \qquad (14)\ (C)$$

In this case we may ignore terms of order $1/r$ in $(\partial N/\partial T)_V$, and so with sufficient accuracy $x \propto T^{\frac{1}{2}}$. Thus

$$\left(\frac{\partial N}{\partial T}\right)_V = \frac{1}{2}\frac{M}{T}\frac{x}{(1+x)^2},$$

$$\frac{C_V}{(N+M)k} = \frac{3}{2} + 12\left(\frac{1}{\beta}-1\right) + \frac{1}{3}\frac{M}{N+M}\frac{x}{(1+x)^2}\left(\frac{3}{2}\right)^2 \qquad (17)\ (C)$$

§ 5. *Calculations of* C_P.—For the total heat I of the assembly we have

$$I = E + PV = E + (N+M)kT + \tfrac{1}{3}aVT^4.$$

The simplest course is to express V and the equation of dissociative equilibrium for N in terms of P_G (not P), and so we shall calculate C_P as

$$\left(\frac{\partial I}{\partial T}\right)_{P_G} + \left(\frac{\partial I}{\partial P_G}\right)_T \left(\frac{\partial P_G}{\partial T}\right)_P - \left(\frac{\partial I}{\partial T}\right)_{P_G} - \frac{4}{3}aT^3\left(\frac{\partial I}{\partial P_G}\right)_T.$$

In case *A*

$$I = \tfrac{5}{2}(N+M)kT - M(\chi_{r+1}+\chi_{r+2}+ \ldots +\chi_{z-1}) - M_1\chi_r + \tfrac{4}{3}a(N+M)kT^5/P_G,$$

$$C_P = \tfrac{5}{2}(N+M)k + \frac{20}{3}a(N+M)kT^4/P_G + \frac{16}{9}a^2(N+M)kT^8/P_G^2 +$$

$$+\left(\tfrac{5}{2}kT + \chi_r + \tfrac{4}{3}akT^5/P_G\right)\left\{\left(\frac{\partial N}{\partial T}\right)_{P_G} - \frac{4}{3}aT^3\left(\frac{\partial N}{\partial P_G}\right)_T\right\}. \qquad (18)\ (A, B)$$

In case *B* there is the same equation for C_P with χ_{r+1} instead of χ_r.

For calculating the variations of N we use, instead of (5),

$$\frac{kTK(T)}{P_G} = \frac{N}{N+M}\frac{M_2}{M_1} = x\left[\frac{r+\dfrac{x}{1+x}}{r+1+\dfrac{1}{1+x}}\right].$$

In case *A* therefore (and in case *B* with χ_{r+1} for χ_r),

$$\frac{dx}{x}\left(1+O\left(\frac{1}{r^2}\right)\right)^* = \left(\tfrac{5}{2}+\frac{\chi_r}{kT}\right)\frac{dT}{T} - \frac{dP_G}{P_G} \qquad (19)\ (A, B)$$

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* $O\left(\frac{1}{r^2}\right)$ denotes "terms of order $1/r^2$."

With the help of equation (10) we therefore find that

$$\frac{C_P}{(N+M)k}=\frac{5}{2}+20\left(\frac{1}{\beta}-1\right)+16\left(\frac{1}{\beta}-1\right)^2+\frac{M}{N+M}\frac{x}{(1+x)^2}\times$$
$$\times\left\{\frac{5}{2}+\frac{\chi_r}{kT}+4\left(\frac{1}{\beta}-1\right)\right\}^2\left(1+O\left(\frac{1}{r^2}\right)\right) \quad . \quad (20)\ (A, B)$$

Similar arguments in case *C* show that

$$C_P=\frac{5}{2}(N+M)k+\frac{20}{3}a(N+M)kT^4/P_G+\frac{16}{9}a^2(N+M)kT^8/P_G^2+$$
$$+\left(\frac{5}{2}kT+\frac{4}{3}akT^5/P_G\right)\left\{\left(\frac{\partial N}{\partial T}\right)_{P_G}-\frac{4}{3}aT^3\left(\frac{\partial N}{\partial P_G}\right)_T\right\}, \quad (18)\ (C)$$

$$\frac{dx}{x}\left(1+O\left(\frac{1}{r^2}\right)\right)=\frac{1}{3}\left\{\frac{5}{2}\frac{dT}{T}-\frac{dP_G}{P_G}\right\}, \quad (19)\ (C)$$

and finally

$$\frac{C_P}{(N+M)k}=\frac{5}{2}+20\left(\frac{1}{\beta}-1\right)+16\left(\frac{1}{\beta}-1\right)^2+\frac{1}{3}\frac{M}{N+M}\frac{x}{(1+x)^2}\times$$
$$\times\left\{\frac{5}{2}+4\left(\frac{1}{\beta}-1\right)\right\}^2\left(1+O\left(\frac{1}{r^2}\right)\right) \quad . \quad (20)\ (C)$$

If the change of dissociation is ignored, these formulæ lead to a value of γ in agreement with Eddington's formula for the case in which the ratio of the material specific heats is 5/3.

§ 6. *Cases of more than Two Important States.*—As mentioned above, in general at least 90 per cent., and indeed usually 95 per cent., of the atoms of a given element are in one of two successive states of ionisation. It does, however, happen in certain small ranges of temperature and density that this is not so, but that to cover 95 per cent. three or more successive states must be included. This occurs only when the χ's corresponding to the two successive changes are almost equal. It should, therefore, be possible to use the above formulæ with an intermediate value for χ, provided we choose an appropriate value for $x/(1+x)^2$. Now if we neglect terms of order $1/r^2$—an approximation one stage less accurate than the foregoing work—it is easy to obtain new formulæ taking into account three successive states. If the numbers of atoms in the three states are M_{r+1}, M_r, and M_{r-1}, and we write $M_{r+1}/M_r=x$, $M_{r-1}/M_r=y$, then, instead of (4) and (10), we find

$$N=M\left\{r+\frac{x-y}{1+x+y}\right\} \qquad dN=M\frac{(1+2y)dx-(1+2x)dy}{(1+x+y)^2} \quad . \quad (21)$$

By using these we obtain formulæ for the isothermals, C_V and C_P, in terms of x, y, and the two χ's. If we identify the two χ's and compare the new formulæ with the corresponding terms in those for two states, we find that the only change is that $x/(1+x)^2$ is replaced by

$$\frac{x+y+4xy}{(1+x+y)^2} \quad . \quad . \quad . \quad . \quad (22)$$

In using our original formulæ we have made this substitution not only in these terms, of order $1/r$, but also in the small correction terms of order $1/r^2$; the error involved must be very small.

In a similar manner our formulæ can be adapted to apply to any number of states. For example, if there are five states of importance, and the numbers of atoms in these are M_{r+2}, M_{r+1}, M_r, M_{r-1}, M_{r-2} respectively, and we write $M_{r+2}/M_{r+1}=x_2$, $M_{r+1}/M_r=x_1$, $M_{r-2}/M_r=y_1$, $M_{r-2}/M_{r-1}=y_2$, then instead of (4)

$$N=Mr\left\{1+\frac{1}{r}\,\frac{x_1(1+2x_2)-y_1(1+2y_2)}{1+x_1(1+2x_2)+y_1(1+2y_2)}\right\}. \qquad (23)$$

and we find that $x/(1+x)^2$ is properly replaced by

$$\frac{x_1(1+4x_2)+y_1(1+4y_2)+4x_1y_1(1+3x_2+3y_2+8x_2y_2)}{\{1+x_1(1+2x_2)+y_1(1+2y_2)\}^2}. \qquad (24)$$

We actually had to make use of this in one instance, but we never found a case where 5 states did not include at least 97 per cent. of the atoms. The adaptation of our formulæ to more than 5 states, though quite simple, is therefore without practical interest.

§ 7. *Numerical Results.*

Table I. gives the results obtained by application of our formulæ arranged in order of decreasing temperature.

Examples 4, 7, 9, 14 refer to Eddington's latest values * for the central conditions of the four Cepheids which he considers behave most regularly.

Examples 6, 10, 17, 20 are derived from the above by multiplying the temperature by 2/3, and the density by $(2/3)^3$, and should represent the average conditions for the same four stars.

Example 8 represents the conditions at the centre of a typical giant, of mass $1\frac{1}{2}$ times that of the Sun.

Examples 5, 3, 2, 1 are for stars of the same mass as the typical giant, but of smaller radius.

Examples 12, 19, 21, 22 are for various distances from the centre of the typical giant.

Examples 11, 12, 13, and similarly 15, 16, 17, 18 show the effect of change of density at constant temperature.

Examples 4A and 21A refer to two stars of mass $19{\cdot}6\odot$, one dense and the other very diffuse. They represent the typical Eclipsing Variable and Cepheid of Eddington, table vii. (*M.N.*, **84**, p. 324).

Further details of these selections will be found in our former paper.

The column headed $\mu=2{\cdot}2$ refers to an ideal non-dissociating monatomic gas of molecular weight 2·2. For such a gas γ depends only on $(1/\beta-1)$, the ratio of the radiation pressure to the material pressure; it is therefore constant throughout any star or series of stars of the same mass. For the standard giant and other giants of the same mass this value is 1·54; for our examples of Cepheids it has a slightly lower value because these stars have greater masses. For

* By letter.

TABLE I.

Values of γ.

Star or Type of Star to which Calculations refer.	T °K.	ρ. gm./c.c.	γ $\mu=2\cdot2$.	γ H.	γ O.	γ Fe.	γ Zn.	γ Br.	γ Ag.
1. Standard mass ; radius $\frac{1}{16}$ standard	$1\cdot054\times10^8$	444	1·535	1·62	..	..	..	..	1·53*
2. Standard mass ; radius $\frac{1}{8}$ standard	$5\cdot27\times10^7$	55·4	1·535	1·62	1·545	1·54*	..	..	1·52*
3. Standard mass ; radius $\frac{1}{4}$ standard	$2\cdot636\times10^7$	6·95	1·54	1·62	1·545	1·52*	..	..	1·53*
4. R.R. Lyr. Centre	$1\cdot83\times10^7$	1·18	1·48	1·585	1·505	1·45	..	..	1·485
4A. Star *g*. Typical eclipsing variable	$1\cdot53\times10^7$	0·155	1·41	1·49	1·39	1·38	..	..	1·405
5. Standard mass ; radius $\frac{1}{2}$ standard	$1\cdot318\times10^7$	0·867	1·54	1·62	1·56	1·46	..	..	1·51
6. R.R. Lyr. Mean	$1\cdot22\times10^7$	0·372	1·48	1·59	1·50	1·43	..	..	1·46
7. R.T. Aur. Centre	$8\cdot76\times10^6$	0·0657	1·45	1·545	1·46	1·40	..	..	1·335
8. Standard Giant. Mass $1\frac{1}{2}$ times Sun	$6\cdot59\times10^6$	0·1085	1·54	1·62	1·55	1·54	1·525	1·49	1·45
9. η Aqu. Centre	$6\cdot00\times10^6$	0·015	1·43	1·525	1·44	1·43	..	..	1·365
10. R.T. Aur. Mean	$5\cdot85\times10^6$	0·0208	1·45	1·55	1·46	1·45	..	..	1·375
11.	$5\cdot64\times10^6$	4·06	1·665	1·665	..	1·61	..	..	..
12. Standard giant ; central distance $r_0/6\cdot9$	$5\cdot64\times10^6$	0·0678	1·54	1·62	..	1·53	..	..	..
13.	$5\cdot64\times10^6$	0·0004	1·34	1·36	..	1·34	..	..	..
14. Y Oph. Centre	$4\cdot24\times10^6$	0·0029	1·40	1·48	1·41	1·395	..	..	1·38
15.	$4\cdot00\times10^6$	0·528	1·65	1·66	..	..	..	1·475	..
16.	$4\cdot00\times10^6$	0·0506	1·58	1·64	..	..	..	1·35	..
17. η Aqu. Mean	$4\cdot00\times10^6$	0·0047	1·435	1·53	1·45	1·43	1·42	1·285	1·41
18.	$4\cdot00\times10^6$	0·00042	1·35	1·39	..	..	..	1·335	..
19. Standard giant ; central distance $2r_0/6\cdot9$	$3\cdot84\times10^6$	0·0215	1·54	1·62	1·54	1·445	1·37	1·33	1·475
20. Y Oph. Mean	$2\cdot82\times10^6$	0·0009	1·405	1·485	1·405	1·395	1·32	1·31	1·375
21. Standard giant ; central distance $3r_0/6\cdot9$	$2\cdot37\times10^6$	0·0050	1·54	1·62	1·55	1·315	1·345	[1·45]	1·38
21A. Star *f*. Typical cepheid	$2\cdot00\times10^6$	0·00033	1·40	1·50	1·42	1·315	..	..	1·32
22. Standard giant ; central distance $4r_0/6\cdot9$	$1\cdot38\times10^6$	0·0010	1·54	1·62	1·535	1·30	[1·45]	..	[<1·33]

* The correction taking account of electrostatic charges has been investigated for these examples but not applied ; it may possibly raise the values for iron by ·02 and for silver by ·03. At lower densities it is negligible.

The values enclosed in brackets [ ] may be as much as ·05 in error. The conditions in these cases are intermediate between those assumed in case *B* and in case *C*. The two terms in the formula for γ are of comparable size and neither may be neglected

exceptionally high densities the value of γ is varied towards 1·67 ; for exceptionally low densities the value of γ is lowered towards 1·33.

The values of γ for hydrogen are of course appreciably higher because the mean molecular weight being only 0·5 the ratio $(1/\beta - 1)$ of radiation pressure to material pressure for given temperature and density is over 4 times smaller.

For the other elements the calculated values of γ generally differ only slightly from those for the ideal non-dissociating gas. This is due to the fact that in general the dissociative equilibrium varies only very slowly over large ranges of temperatures and densities. Definite exceptions to this rule for each element occur in ranges where that element is shedding a shell of electrons, generally the L shell. For example :

Iron	is shedding its	L	shell	in	examples	21, 22.
Zinc	,,	,, L	,,	,,		19, 20, 21.
Bromine	,,	,, L	,,	,,		16, 17, 19, 20.
Silver	,,	,, L	,,	,,		7, 8, 9, 10.
,,	,,	,, M	,,	,,		21, 22.

In such cases the dissociative equilibrium varies comparatively rapidly, and γ has low values, usually below 1·4, and sometimes as low as 1·3. Now for a spherical mass of gas to be stable γ must be greater than 1·33, and it is therefore important to examine whether these exceptional ranges of low γ do or do not seriously overlap for different elements. We notice that the critical ranges of iron (Z=26) and bromine (Z=35) are distinct, while the critical range of zinc (N=30) does overlap both that of iron and that of bromine. We may conclude that at the most, for the conditions actually occurring in any star, there may be as many as seven or eight elements (of which possibly all but one or two will be very scarce) simultaneously passing through the critical stage of losing their L shell ; these elements will be those with atomic numbers between 25 and 45. Elements with lower atomic number than 25 are not affected in this way, as even at the lowest temperatures occurring in stellar interiors they have lost their L shell. The same sort of thing happens in the shedding of the M shell, but only in the case of elements with atomic number greater than 40.

We may conclude that unless a star happens to have a composition consisting largely of elements, which just happen to be in a critical stage under the existing conditions, the value for γ is only very slightly less than that for a non-dissociating perfect gas of molecular weight equal to the actual mean molecular weight. That is to say, with the above reservation, the value of γ can generally be calculated without serious error by Eddington's formula

$$\frac{\gamma-\frac{4}{3}}{\Gamma-\frac{4}{3}}=\frac{4-3\beta}{1+12(\Gamma-1)(1-\beta)/\beta} \quad . \quad . \quad . \quad (25)$$

provided we give Γ the value 5/3 corresponding to a non-dissociating perfect gas. It does, however, appear to be the case that for massive, very low-temperature Cepheids typified by 21A of Table I., and to a

less extent Y Oph., the elements near Fe and Ag are both in critical regions, and a definite slight diminution in γ below its non-dissociating value is certainly possible, and might almost be said to be expected. More than this cannot be said without a definite choice of material constitution consisting of a fairly large number of representative elements. Extensions of the foregoing analysis would be required to deal with this case, which could easily be worked out on the same lines.

In conclusion, it must be pointed out that there can be no analogue in the general theory of a dissociating gas for the simple relation (25) between γ, the (P, V)-adiabatic index for the whole assembly, and Γ, the ratio of the *material* specific heats. This is because changes of V and T in the general case affect the degree of dissociation, and so the pressure *via* N, which destroys the simplicity of the relationship. It is, therefore, impossible to make any use of the function Γ at all—only the complete γ is significant.

Notes on the Theory of Absorption Lines in Stellar Spectra.

By R. H. Fowler.

1. This paper contains certain notes additional to and correcting the theory of absorption lines as hitherto presented * by Professor Milne and the present writer. They arise in a more exact formulation of the necessary theorems of statistical mechanics. No attempt is made here to present any deep-going reformulation of the whole theory, such as is required to put the quantitative theory of absorption spectra on a proper footing. Such a reformulation may be expected to develop naturally out of the recent work of Professor Milne.†

The points of this note are taken in the main from a re-discussion of the existing theory which formed one chapter of an Adams Prize Essay for 1923–24 on the Physical State of Matter at High Temperatures. A simultaneous independent re-discussion by Miss C. Payne has just been published in book form.‡ This is a thorough review of existing data, to which the author herself has made large contributions, and it is only necessary here to make points supplementary to her account.

2. *The Form of Partition Functions for Excited Atoms. Weights and Symmetry Numbers.* It has already been pointed out by Milne § that the complete formula for the fraction of atoms ionised is incorrectly quoted,‖ and should contain another term, $b_1(\mathrm{T})$ or $u_1(\mathrm{T})$, the partition function for the ionised atom, so that only *the ratio* of the two partition functions enters. The calculations of our former papers were therefore carried out on the basis of the assumption that the ionised atoms are

* Fowler and Milne, *M.N.*, (i), **83**, 403 (1923); (ii), **84**, 499 (1924).
† Milne, *M.N.*, **85**, 739, 750, 768 (1925).
‡ Payne, "Stellar Atmospheres," *Har. Obs. Monographs*, No. 1 (1925).
§ Milne, *Phil. Mag.*, **50**, 547 (1925).
‖ Fowler and Milne, *loc. cit.* (i).

always in their lowest state of weight unity. Further, in the absence of proper information, all weights for all stages were taken to be unity in these calculations. This is not correct, and the proper numerical values have since then become clearly determinable, thanks largely to studies of the Zeeman effect of the more complicated spectra. A full statement of the proper numerical values of the weights for use in any statistical calculation will be found in a forthcoming paper * with references to the original sources. The proper numerical values are emphatically different from unity, but it is just not worth while at this stage to repeat the calculations of our former papers and Miss Payne's book. The values are specified with full regard to the occurrence of groups of equivalent electrons, so that symmetry numbers as such do not enter. Their introduction into the formulæ of atomic dissociation seems now to have been due to an insufficiently analysed analogy with molecular structure.

The neglect, according to Urey's theory, of all but the first few terms in $u(\mathrm{T})$ was justified in the earlier papers.† Since then other theories of the correct form of $u(\mathrm{T})$ have been advanced. These are discussed together in the forthcoming paper already referred to, and it is found that it remains true in all forms of the theory that in reversing layer problems all but the first few terms of $u(\mathrm{T})$ may be neglected, and in general, in fact, all but the first (normal) term and the term to which the lines in question belong.

3. *Decay of Lines past their Maxima.*—It seems possible to explain in general terms the different rates of dying out of the Balmer series of hydrogen and the lines of the ordinary metallic elements as the temperature increases beyond the maximum. The statistical reason for this different behaviour is simple and possibly well known, but I have never seen it stated in print.‡ A line of the Balmer series, or any line of the spectra H I, He II, etc., is a line of the last spectrum the atom can emit, having no further electrons to lose. Spectra such as Si I, Fe I, Si II, etc., belong on the other hand to atoms that can lose a regular sequence of further electrons at not too widely spaced ionisation potentials. A spectrum such as Si IV or Mg II, and perhaps Na I, will be effectively the last spectrum of these atoms, for the next electron is separated by a big step in ionisation potential, and will not become effectively removeable at the temperatures of stellar reversing layers. The numerically different behaviour of "last" and "not last" spectra can be at once established.

Let $x_0, x_1, \ldots$ be the fractions of atoms present, which are neutral, once ionised, . . . , etc. Then §

$$\frac{x_{r+1}}{x_r}\mathrm{P}_e = \frac{(2\pi m)^{\frac{3}{2}}(k\mathrm{T})^{\frac{5}{2}}}{h^3} e^{-\chi_r/k\mathrm{T}} \frac{u_{r+1}(\mathrm{T})}{u_r(\mathrm{T})}. \quad . \quad . \quad . \quad (1)$$

In equation (1) P_e is the partial electron pressure, χ_r is the ionisation energy of the normal already r-times ionised atom, m the mass of the

* Fowler, *Phil. Mag.*, in publication.
† Fowler and Milne, *loc. cit.* (ii), § 9.
‡ I owe my appreciation of this point to a discussion with Professor Bohr.
§ Fowler, *loc. cit.*, in publication ; Milne, *loc. cit.*

electron, and h and k Planck's and Boltzmann's constants. If $(n_r)_s$ is the fraction of all the atoms r-times ionised and in their s^{th} state, then

$$(n_r)_s = \frac{x_r(\varpi_r)_s\, e^{-\{\chi_r-(\chi_r)_s\}/kT}}{u_r(T)} \quad . \quad . \quad . \quad (2)$$

where $\chi_r-(\chi_r)_s$ is the energy of excitation of the s^{th} level. The behaviour of the line of a last spectrum of an r-times ionised atom is determined by (1), (2), and the equation

$$x_r+x_{r+1}=1 \quad . \quad . \quad . \quad . \quad (3)$$

near and on the high-temperature side of its maximum. No other stages of ionisation are appreciably present for any relevant temperature, however high. We thus find, on inserting numerical values,

$$(n_r)_s = \frac{(\varpi_r)_s\, e^{(\chi_r)_s/kT}}{u_r(T)e^{\chi_r/kT}+\dfrac{0{\cdot}332}{P_e}T^{\frac{3}{2}}u_{r+1}(T)} \quad . \quad . \quad (4)$$

This is the complete form of the usual formula, already frequently used to determine maxima.

The high temperature behaviour of a " not last " spectrum, ZI for example, is determined on the contrary by (1), (2), and

$$x_0+x_1 \ldots +x_r+x_{r+1}=1 \quad . \quad . \quad . \quad (5)$$

if the $(r+1)$-times ionised stage is still the last relevant. This leads of course to an expression identical with (4) near the maximum where (5) becomes effectively $x_0+x_1=1$. But at higher temperatures we have effectively $x_1+x_2=1, \ldots$ then $x_r+x_{r+1}=1$, and $(n_0)_s$ will be substantially diminished. The physical reason for this decrease is easily seen. So long as the commonest ion has merely to catch one electron to get into the required state, the chance of doing so diminishes as the temperature increases, it is true, but not excessively rapidly. But if it has to catch two or more electrons, its chance of being in the proper state diminishes more or less like the square or higher power of the former chance. In the case specified we have

$$\frac{x_r}{x_0}P_e{}^r=(0{\cdot}332)^r T^{\frac{3}{2}r} e^{-(\chi_0+\ldots+\chi_{r-1})/kT}\frac{u_r(T)}{u_0(T)},$$

and so

$$(n_0)_s=\frac{x_r}{u_r(T)}\frac{(\varpi_0)_s e^{-\{\chi_0-(\chi_0)_s\}/kT}}{\left(\dfrac{0{\cdot}332}{P_e}\right)^r T^{\frac{3}{2}r} e^{-(\chi_0+\ldots+\chi_{r-1})/kT}} \quad . \quad . \quad (6)$$

This is to be combined with (1) and (3). Thus finally

$$(n_0)_s = \frac{(\varpi_0)_s e^{\{\chi_0-(\chi_0)_s\}/kT}}{u_r(T)+\dfrac{0{\cdot}332}{P_e}T^{\frac{3}{2}}e^{\chi_r/kT}u_{r+1}(T)} \times \frac{1}{\left(\dfrac{0{\cdot}332}{P_e}\right)^r T^{\frac{3}{2}r} e^{-(\chi_0+\ldots+\chi_{r-1})/kT}} \quad (7)$$

Equation (7) holds quite generally for the region of temperature for which (3) is approximately true. It holds also for later spectra than the first if the suffix o is replaced by a and the index r by $r-a$.

Simple calculation for an idealised case is amply sufficient to show the effect of the extra term in (7). Let us compare H I with an idealised Si I of the same ionisation potential and term values ($\chi_0 = 13{\cdot}54$ volts), the following ionisation potentials for Si being (say) 20, 31·5, and 45 volts, and carry through the calculations for a temperature of 25,000° K. and $P_e = 10^{-4}$ atmos., roughly the maximum of the Si IV lines. For simplicity we will ignore differences in the weights and take $(\varpi_0)_s$ and the $u_r(T)$ all unity. Dropping factors the same for both atoms, we find

$$\text{H I} \qquad (n_0)_s \propto \frac{1}{1+\dfrac{0{\cdot}332}{P_e}T^{\frac{5}{2}}e^{-13{\cdot}54/kT}} = 10^{-5{\cdot}78},$$

$$\text{Si I} \qquad (n_0)_s \propto \frac{1}{1+\dfrac{0{\cdot}332}{P_e}T^{\frac{5}{2}}e^{-45/kT}} \times \frac{1}{\left(\dfrac{0{\cdot}332}{P_e}\right)^3 T^{\frac{15}{2}}e^{-65/kT}} = 10^{-12{\cdot}53}.$$

There is thus an extra factor $10^{-6{\cdot}7}$ at 25,000° K., reducing the lines of the idealised Si I compared with those of H I. The fraction of H atoms capable of absorbing the Balmer lines at maximum is about 10^{-5}, reduced by an extra factor of $10^{-1{\cdot}82}$ at 25,000° K.* The reduction factor for similar Si I atoms is therefore $10^{-8{\cdot}5}$.

These purely typical calculations at once explain the complete disappearance of all metallic lines at higher temperatures compared with the striking persistence of H. For in all the metals of the Fe group from Sc to Ni (or even Cu) there is always an adequately long series of ionisation potentials not too far apart, and the foregoing arguments must hold. They will hold also for Si, S, and Al, and to a less extent for O and N. The rate of disappearance of the He I lines will hardly be reinforced in this way before a temperature of 30,000° K. Lines of Ca I and Mg I and similar spectra, followed by one near ionisation potential, will have their rate of decay considerably amplified, though not, of course, to the degree calculated above. The H and K lines of Ca should behave more or less like a last spectrum. There will remain certain cases of fairly rapid disappearance which can hardly be accounted for in this way. An example is given by the lines of Mg II, which are followed by an ionisation potential of the order of 60 volts.

It is clear that these considerations will tend to bring the appearance and disappearance points of the lines of numerous spectra into much better theoretical agreement. They have also a close connection with the considerations of relative abundance put forward by Miss Payne,† in so far as these are based on points of disappearance. It seems likely that a detailed re-discussion of fractional concentrations $(n_r)_s$ and relative abundances would be of value.

* Fowler and Milne, *loc. cit.* (i), fig.

† Payne, *loc. cit.*, chap. xiii.

4. *Effects of u(T) on the Positions of Maxima.*—Applications of the theory have been hitherto made, ignoring the differential coefficients of the u(T). This is usually correct, but in certain cases the terms of low excitation energy may make important contributions, which are easily calculated.

Using equation (4), the complete relation between P_e and T_{max} for a line absorbed by a neutral atom in its s^{th} state is easily found to be

$$P_e = 0{\cdot}332\frac{u_1(T)}{u_0(T)}\left\{\frac{(\chi_0)_s+\frac{5}{2}kT+kT^2u_1'/u_1}{\chi_0-(\chi_0)_s-kT^2u_0'/u_0}\right\}T^{\frac{5}{2}}e^{-\chi_0/kT} \quad . \quad (8)$$

The total number of neutral atoms at this maximum is given by

$$(x_0)_{max}=\frac{(\chi_0)_s+\frac{5}{2}kT+kT^2u_1'/u_1}{\chi_0+\frac{5}{2}kT+kT^2(u_1'/u_1-u_0'/u_0)} \quad . \quad . \quad (9)$$

The same formulæ hold for the lines of any state of the r-times ionised atom, with change of suffixes, when all stages of ionisation except r and $r+1$ can be ignored.

Suppose for simplicity that

$$u(T)=\varpi_0+(\varpi_0)_1e^{-\{\chi_0-(\chi_0)_1\}/kT}.$$

Then the denominator of P_e becomes

$$\varpi_0\{\chi_0-(\chi_0)_s\}+(\varpi_0)_1\{(\chi_0)_1-(\chi_0)_s\}e^{-\{\chi_0-(\chi_0)_1\}/kT},$$

the second term being representative of a series of similar terms. Unless $\chi_0-(\chi_0)_1$ is small compared to kT, these extra terms can be properly ignored. If $(\chi_0)_1$ and $(\chi_0)_s$ are equal, or nearly equal, the corresponding term is in any case negligible. We can, in fact, for practical purposes group together with the normal term all terms, of total weight $[\varpi_0]$, for which $\chi_0-(\chi_0)_1$ is small compared both to kT and $\chi_0-(\chi_0)_s$, and take the denominator of P_e to be $[\varpi_0]\{\chi_0-(\chi_0)_s\}$. No effect of this sort can be produced by u_1'/u_1, which can in general be ignored. We can therefore write

$$P_e=0{\cdot}332\frac{u_1(T)}{[\varpi_0]}\frac{(\chi_0)_s+\frac{5}{2}kT}{\chi_0-(\chi_0)_s}T^{\frac{5}{2}}e^{-\chi_0/kT}, \quad . \quad . \quad (10)$$

In this case $[\varpi_0]$ may be appreciably different from u_0(T) (see (8)), which it replaces when u_0'(T) is not ignored. In calculating $(x_0)_{max}$, both u_1' and u_0' can, in general, be neglected.

The terms in u'(T) may also be important in maxima of ultimate lines of an ionised atom. Here we retain three stages. Then with two equations (1) we combine

$$x_0+x_1+x_2=1, \quad n_1=\varpi_1x_1/u_1(T).$$

On solving for n_1, the fraction of atoms once ionised and in their normal state, we find

$$n_1=\frac{\varpi_1}{u_1(T)+u_0(T)\frac{P_e}{0{\cdot}332}T^{-\frac{5}{2}}e^{\chi_0/kT}+u_2(T)\frac{0{\cdot}332}{P_e}T^{\frac{5}{2}}e^{-\chi_1/kT}} \quad . \quad (11)$$

The maximum of n_1 will occur when

$$kT^2u_1'(T)-\frac{P_e}{0{\cdot}332}\{\chi_0+\tfrac{5}{2}kT-kT^2u_0'/u_0\}u_0(T)T^{-\frac{5}{2}}e^{\chi_0/kT}$$
$$+\frac{0{\cdot}332}{P_e}\{\chi_1+\tfrac{5}{2}kT+kT^2u_2'/u_2\}u_2(T)T^{\frac{5}{2}}e^{-\chi_1/kT}=0 \qquad (12)$$

If we can ignore $u_1'(T)$, we then find

$$P_e=0{\cdot}332\left\{\frac{\chi_1+\frac{5}{2}kT+kT^2u_2'/u_2}{\chi_0+\frac{5}{2}kT-kT^2u_0'/u_0}\frac{u_2(T)}{u_0(T)}\right\}^{\frac{1}{2}}T^{\frac{5}{2}}e^{-\frac{1}{2}(\chi_0+\chi_1)/kT} \qquad (13)$$

which is the complete form of the usual formula.* The order of the two terms retained in (12) is $(u_0u_2)^{\frac{1}{2}}e^{-\frac{1}{2}(\chi_1-\chi_0)/kT}$ at the maximum, and the order of the neglected term is $u_1e^{-(\chi_1-(\chi_1)_1)/kT}$. This term, however, can be of the same or even higher order (the principal lines of Ca II, Sr II, and Ba II are examples of the latter). The condition is the existence of a state of low excitation energy in the singly ionised spectrum. When this term is of higher order, the third term can be ignored, and we have, putting $u_1(T)=\varpi_1+(\varpi_1)_1e^{-\{\chi_1-(\chi_1)_1\}/kT}$,

$$P_e=0{\cdot}332\frac{(\varpi_1)_1\{\chi_1-(\chi_1)_1\}}{u_0(T)\{\chi_0+\frac{5}{2}kT-kT^2u_0'/u_0\}}T^{\frac{5}{2}}e^{-\{\chi_0+\chi_1-(\chi_1)_1\}/kT} \quad . \qquad (14)$$

In both cases $(n_1)_{max}=\varpi_1/u_1(T)=1$ approximately, and u_0' and u_2' can in general be ignored in both (13) and (14). The maximum is determined in the second case by the change over of normal ionised atoms into their first excited state instead of as usual into the next stage of ionisation.

In conclusion one may note that no general discussion of $u'(T)$ can be expected to cover all cases, and it is always simple to form an explicit 2- or 3-term formula for the relevant part of $u(T)$, and so calculate $u'(T)$ exactly.

5. *Calculations.*—Miss Payne † has examined the suggestion that the product $P_e[(n_r)_s]_{max}$ should be approximately the same for any line of any atom. She has shown that in the lower temperature stars, for which fairly good independent determinations of temperature are available, this suggestion is in good agreement with the facts. The constant value is about 10^{-10} (P_e in atmospheres). This suggestion is a natural extension of the discussion of the value of P_e for principal lines.‡ There are good theoretical grounds for regarding it as a valuable approximation to the truth. The present writer has independently analysed the data § on the same assumption, and come to the same conclusion with a constant 2×10^{-10}, using the latest values for all weights, and the improvements in theory of § 4. In general it may be said that the maxima of ordinary subordinate lines of excitation potential one-half or so of the ionisation potential, principal (ultimate) lines, and the nearly ultimate lines of very small excitation potential so common in the

* Fowler and Milne, *loc. cit.* (ii), equation (3).
† *Stellar Atmospheres*, chap. ix.
‡ Fowler and Milne, *loc. cit.* (ii), § 7.
§ Menzel, *Har. Coll. Obs. Circ.*, No. 258; Payne, *ibid.*, Nos. 252, 256 (1924).

Fe group all fit (with very few exceptions) into a consistent reliable temperature scale.

In view of this agreement it is unnecessary to repeat all the details here, but, owing to the previous neglect of $u'(T)$, all previous calculations for Ca^+, Sr^+, and Ba^+ are to some extent in error, and they are repeated here. The first two terms of (12) are dominant, and formula (14) must be used. Variations of $u_0(T)$ and $u_2(T)$ may be ignored.

(i.) Ba^+. *Principal Doublet.* $\lambda\lambda 4934, 4554$; $\chi_0=5{\cdot}18$, $\chi_1=9{\cdot}95$; $\frac{1}{2}(\chi_1-\chi_0)=2{\cdot}385>1{\cdot}512=\chi_1-(\chi_1)_1$.

$$u_0(T)=\tfrac{1}{4}\ ;\ u_1(T)=\tfrac{1}{2}(2+10e^{-1{\cdot}512/kT})\ ;\ u_2(T)=1.$$

The observed maximum lies at Ma, 3000° K. (Payne), or perhaps lower (M3 or below Menzel).

$T_{max.}$	$\frac{P_e}{\text{atmos.}}$; $(n_1)_{max}=1.$
2500	$1{\cdot}66\times10^{-11}$
3000	$4{\cdot}6\ \times10^{-9}$
3500	$2{\cdot}72\times10^{-7}$

(ii.) Sr^+. *Principal Doublets.* $\lambda\lambda 4216, 4078$; $\chi_0=5{\cdot}665$, $\chi_1=10{\cdot}97$; $\frac{1}{2}(\chi_1-\chi_0)=2{\cdot}652>1{\cdot}815=\chi_1-(\chi_1)_1$.

$$u_0(T)=\tfrac{1}{4}\ ;\ u_1(T)=\tfrac{1}{2}(2+10e^{-1{\cdot}815/kT})\ ;\ u_2(T)=1.$$

The observed maximum lies at K2, 3500° K. (Payne). [Menzel gives Mo, 3000° K.]

$T_{max.}$	$\frac{P_e}{\text{atmos.}}$; $(n_1)_{max}=1.$
3000	$2{\cdot}42\times10^{-10}$
3500	$2{\cdot}02\times10^{-8}$
4000	$6{\cdot}8\ \times10^{-7}$

(iii.) Ca^+. *Principal Doublets.* $\lambda\lambda 3968, 3934$; $\chi_0=6{\cdot}08$, $\chi_1=11{\cdot}82$; $\frac{1}{2}(\chi_0-\chi_1)=2{\cdot}87>1{\cdot}69=\chi_1-(\chi_1)_1$.

$$u_0(T)=\tfrac{1}{4}\ ;\ u_1(T)=\tfrac{1}{2}(2+10e^{-1{\cdot}69/kT})\ ;\ u_2(T)=1.$$

The observed maximum is at K0, 4000° K.

$T_{max.}$	$\frac{P_e}{\text{atmos.}}$; $(n_1)_{max}=1.$
3500	$7{\cdot}4\ \times10^{-9}$
4000	$2{\cdot}56\times10^{-7}$
4500	$3{\cdot}9\ \times10^{-6}$

It will be observed that, as now calculated, these maxima are somewhat out of step with the general body of other maxima for which $P_e n_{max}=2\times10^{-10}$. The calculated maxima occur at *lower* temperatures than the observed about 500° for Sr^+ which is the best determined, more for Ca^+, and less for Ba^+. It seems almost certain that there is a significant connection between this fact and the fact that these are all

principal lines. The new theory has as it were over-corrected the old. It is tempting to suggest that these delayed maxima are due to the increasing chromospheric absorption of these lines in higher temperature stars.* The same effects may perhaps be expected for lines arising from *metastable* states of very low excitation energy in metals of the Fe group. This must await examination in a more precise theory.

6. *Summary.*—Previous formulæ for the maxima of stellar absorption lines are given in a more exact form.

The theoretical positions of the maxima of the principal lines of Ca^+, Sr^+, and Ba^+ are recalculated in connection with the recent work of Miss Payne and are shown now to be at somewhat *lower* temperatures than the observations indicate. A reason for this is suggested.

The theoretical rate of decay of lines on the high-temperature side of their maxima is re-investigated. It is shown that the striking observed difference between the Balmer lines of hydrogen and the lines of the metals of the Fe group (and others) is in full agreement with the theory.

Ionisation in Stellar Atmospheres and Steric Factor: a Reply to Mr. M. C. Johnson. By Megh Nad Saha.

In a paper entitled "Cumulative Ionisation in Stellar Atmospheres,"† Mr. Martin C. Johnson has criticised certain views of the present writer regarding the so-called "steric factor." On going through the matter in detail, I find that his criticisms cannot be upheld. I shall take his criticisms one by one.

Objection 1 (in Mr. Johnson's words):

"The principal difficulty in Saha's theory of the steric factor is that it applies to all forms of ionisation equally due to radiation, thermal energy, and applied field energy; it should, therefore, lead to all experimental values of ionisation potentials being lower than the values deduced from convergence frequencies of spectra in the case of most elements. The tabulations of data (*e.g.* Foote and Mohler) show no such discrepancies that could be interpreted as more than naturally distributed errors."

In the electrical experiments on ionisation potential the vapour is bombarded by electrons subjected to a gradually increasing voltage, and the current voltage curve plotted (after due precautions). The particular voltage at which a sudden increase occurs in the current is taken as the ionisation potential. Whatever the steric factor may be, the vapour cannot be ionised, and no sudden kick will occur in the curve as long as the E.M.F. impressed on the electrons do not reach the required value, *i.e.* the ionisation potential. For an element

* In Milne's theory of the Ca^+ (or Sr^+, Ba^+) chromosphere, the residual central intensity in the line is fixed by g, τ, and m, and is independent of the photospheric temperature T. As T increases the observed strength of the chromospheric absorption line must therefore increase.

† *M.N.*, **84**, 516, 1924.

having a large steric factor the current at this point may be larger, but the position of the kick will remain unchanged.

Hence there is no reason why the electrical experiments should give a lower value of the ionisation potential for elements with large steric factors.

Objection 2 :

" Saha gives the steric factor large for C and small for Mg. The I.P. of Mg is 7·65 ; for C it is quoted by Professor Fowler as from 7 to 8 volts, on a suggestion due to Saunders. On the theory of the steric factor, C^+ should appear at a lower temperature than Mg^+ in a stellar sequence; but whereas Mg^+ appears in Go stars, C^+ does not appear till B6 or B8."

Even assuming that C has the same I.P. as Mg, which appears to me rather doubtful, it appears that Mr. Johnson has overlooked another point of great importance. Mg^+ is detected in stellar spectra by the line $\lambda=4481$, which corresponds to the combination $(3d-4f)$. Now, in order that an Mg^+ atom may absorb this line, it must be brought from the 1S stage to $3d$ stage, which means an additional potential of 9 volts (15·02, I.P. of $Mg^+-6·1$, the voltage corresponding to $3d$ term of Mg^+, which, according to Fowler, is 49776). But in the corresponding case C^+, the element is detected by the line $\lambda=4267$, which, according to Fowler, belongs to $3d-mf$ combination. In order that C^+ may be enabled to absorb this line, it must be brought from the normal state of C^+ to the $3d$ stage. According to the figures given by Fowler this corresponds to a potential of $24·28-6·29=17·99$ volts. This is just double the corresponding excitation voltage of Mg^+. Thus this objection of Mr. Johnson seems to be quite invalid.

Objection 3 :

" Again, N, an element with a high steric factor, according to Saha, persists as absorption lines of the neutral atom as far as stars of the class Oe5, with the maximum at B2, C^+ having a maximum at the same place in the stellar sequence and beginning and persisting similarly. Now the I.P. of N is variously estimated at from 17 to 30 volts, in any case less than the second I.P. of C^+, which is given by Professor Fowler as $24·3+7$ or 8 volts. But on Saha's theory the steric factor reducing the effective ionisation for N is greater than for C, *i.e.* N should be more easily ionised even if the ionisation potentials were the same."

Probably Mr. Johnson has the line $\lambda=3995$ in mind which, up to this time, was attributed to neutral nitrogen. But in my paper (*Phil. Mag.*, **44**, 1138, 1922) I pointed out that since these groups of lines do not occur in classes cooler than Ao, they are most probably due to N^+.

This prediction has been verified by Fowler's * recent work on the spectrum of ionised nitrogen, and he has now definitely assigned $\lambda=3995$ and other associated lines to N^+. The I.P. of neutral nitrogen is still unknown, but recent work by Kiess † indicates that it is in the neighbourhood of 12 volts. The I.P. of N^+ is also unknown, but Fowler estimates it to be about 24 volts, so that the excitation potential for

* Fowler, *Proc. Roy. Soc. Lond.*, **107** (1925).

† Kiess, *Journ. Am. Optical Society*, 1925 June.

the line $\lambda=3995$ is $12+24-8{\cdot}5=27{\cdot}5$ volts, while the corresponding quantity for the C^+ line $\lambda=4267$ is $7+24-6{\cdot}1=24{\cdot}9$ volts. Thus the nitrogen lines present not only no difficulty to the theory of steric factor, but they probably justify the theory. We must, however, wait till the I.P. of C, N, and N^+ are accurately determined.

Physics Department,
Allahabad University:
1925 *August* 14.

Note on Rosseland's Integral for the Stellar Absorption Coefficient.

By Professor E. A. Milne.

In a recent paper * Rosseland has shown that if $k(\nu)$ is the mass absorption coefficient of a gas in frequency ν, the correct mean absorption coefficient k to use in calculating the net flux of radiation is given by the formula

$$\frac{1}{k}=\int_0^\infty \frac{1}{k(\nu)}\phi_\nu d\nu \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\phi_\nu=\frac{\partial B_\nu}{\partial T}\Big/\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu$$

and

$$B_\nu=\frac{2h\nu^3/c^2}{e^{h\nu/kT}-1},$$

in the usual notation. For brevity we shall call the mean value k derived from $k(\nu)$ in this way the " Rosseland mean-value " of $k(\nu)$.

When a theoretical calculation of k is being undertaken, from atomic theory, it is usual to assume that the gas is made up of one particular element, *e.g.* iron, and to determine k for pure iron vapour. It is then natural to suppose that a mixture of elements will have an absorption coefficient which is a mean amongst the calculated absorption coefficients of the separate constituents.

This, however, is not necessarily true. It is partly the object of this note to point out that when the Rosseland mean absorption coefficient is calculated for each constituent separately, the Rosseland mean absorption coefficient for the mixture in general exceeds that formed by combining the separate Rosseland means according to the proportions in which the constituents are present, and, moreover, will not even necessarily lie between the greatest and least of these.

Let $a_1, \ldots a_n$ be the proportions by mass of the separate constituents, $k_1(\nu), \ldots k_n(\nu)$ their absorption coefficients. Then each constituent has a separate Rosseland mean of the type

$$\frac{1}{k_r}=\int\frac{1}{k_r(\nu)}\phi_\nu d\nu,$$

---

* *M.N.*, **84**, 525, 1924.

whilst the mixture has an absorption coefficient $k(\nu)$ for frequency ν given by

$$k(\nu)=a_1k_1(\nu)+\ \ldots\ +a_nk_n(\nu).$$

The first result stated above follows immediately from the inequality (see Appendix),

$$\frac{1}{\int_0^\infty \frac{\phi_\nu d\nu}{a_1k_1(\nu)+\ \ldots\ +a_nk_n(\nu)}} \geqslant \frac{a_1}{\int_0^\infty \frac{\phi_\nu d\nu}{k_1(\nu)}}+\ \ldots\ +\frac{a_n}{\int_0^\infty \frac{\phi_\nu d\nu}{k_n(\nu)}} \qquad (2)$$

which shows that

$$k\geqslant a_1k_1+\ \ldots\ +a_nk_n.$$

The inequality becomes an equality only when $k_1(\nu),\ \ldots\ k_n(\nu)$ are all proportional to one another for all ν, *i.e.* are the same function except for a multiplying constant.

The second of the results enumerated above can be seen from an example. We take an artificial case. Determine the frequency ν_1 for which

$$\int_0^{\nu_1}\phi_\nu d\nu=\int_{\nu_1}^\infty \phi_\nu d\nu=\tfrac{1}{2}.$$

Suppose there are two constituents. Let the absorption coefficients be given by the relations

$$\begin{array}{llll} k_1(\nu)=\epsilon, & k_2(\nu)=\mathrm{A}, & (0<\nu<\nu_1) \\ k_1(\nu)=\mathrm{A}, & k_2(\nu)=\epsilon, & (\nu_1<\nu<\infty), \end{array}$$

where ϵ and A are constants. It is clear that a mixture of equal masses of the two will have an absorption coefficient $k(\nu)$ given by

$$k(\nu)=\tfrac{1}{2}(\epsilon+\mathrm{A}),$$

constant throughout the spectrum. The Rosseland mean of this is, of course, $k=\frac{1}{2}(\epsilon+\mathrm{A})$ also. But the Rosseland mean for either constituent by itself is given by

$$\frac{1}{k_1}=\frac{1}{k_2}=\frac{1}{2}\left(\frac{1}{\epsilon}+\frac{1}{\mathrm{A}}\right),$$

or

$$k_1=k_2=2\epsilon\mathrm{A}/(\epsilon+\mathrm{A}).$$

Since

$$\tfrac{1}{2}(\epsilon+\mathrm{A})>\frac{2\epsilon\mathrm{A}}{\epsilon+\mathrm{A}},$$

we have not only

$$k>\tfrac{1}{2}(k_1+k_2),$$

in accordance with (2), but also

$$\begin{array}{l} k>k_1, \\ k>k_2. \end{array}$$

The Rosseland mean absorption coefficient of this mixture not only exceeds the mean of the Rosseland means of the separate constituents,

but exceeds either separately. If ϵ is small compared with A, we have $k_1=k_2=2\epsilon$, approximately, whilst $k=\frac{1}{2}$A. The ratio is $\frac{1}{4}$A/ϵ, which may be as large as we please.

We see that when Rosseland means for pure elements are used for a mixture, there is danger of getting too small a coefficient. The Rosseland mean emphasises the smallest values of the absorption coefficient whose mean is being taken. It is the smallest values that have a predominant influence on the mean. But in a mixture, if the small values for one constituent occur in the same spectral region as the large values for another, the small values lose all their importance, and the absorption coefficients calculated for the separate constituents lose most of their significance as regards the absorption coefficient of the mixture.

Suppose, on the other hand, we consider what may be called the arithmetic mean absorption coefficient, k', defined by

$$k'=\int_0^\infty k(\nu)\phi_\nu d\nu.$$

We have in all cases

$$k'=a_1k_1'+\ \ldots\ +a_nk_n'.$$

In consequence of the inequality

$$\int_0^\infty k(\nu)\,\phi_\nu d\nu \geqslant 1 \Big/ \int_0^\infty \frac{1}{k(\nu)}\phi_\nu d\nu \quad . \quad . \quad . \quad (3)$$

which holds in virtue of Schwarz's Inequality, since

$$\int_0^\infty \phi_\nu d\nu=1,$$

we see that

$$k'\geqslant k.$$

Rosseland and Eddington have drawn attention to this result, namely, that the arithmetic mean gives in general too large values. The inequality becomes an equality only if $k(\nu)$ is a constant independent of ν.

For a mixture, in the proportions a_1, . . . a_n, our inequalities give altogether

$$\int_0^\infty [a_1k_1(\nu)+\ \ldots\ +a_nk_n(\nu)]\phi_\nu d\nu$$

$$\geqslant \frac{1}{\displaystyle\int_0^\infty \frac{\phi_\nu d\nu}{a_1k_1(\nu)+\ \ldots\ +a_nk_n(\nu)}} \geqslant \frac{a_1}{\displaystyle\int_0^\infty \frac{\phi_\nu d\nu}{k_1(\nu)}}+\ \ldots\ +\frac{a_n}{\displaystyle\int_0^\infty \frac{\phi_\nu d\nu}{k_n(\nu)}} \qquad (4)$$

i.e.

$$a_1k_1'+\ \ldots\ +a_nk_n'\geqslant k\geqslant a_1k_1+\ \ldots\ +a_nk_n.$$

We now observe that if it happens that the absorption coefficient of the whole mixture,

$$k(\nu)=a_1k_1(\nu)+\ \ldots\ +a_nk_n(\nu),$$

considered as a function of ν, is a constant independent of ν, whilst the same is not true of $k_1(\nu)$, . . . $k_n(\nu)$ separately, then not only will the arithmetic mean be equal to the Rosseland mean but (1) the arithmetic means of the separate constituents combined in the proportions a_1, . . . a_n will give the correct value of the coefficient for the mixture, namely, the common value of the arithmetic mean and the Rosseland mean for the mixture; (2) the same will not be true of the Rosseland means of the separate constituents, which, when combined in the proportions a_1, . . . a_n, will give too small a value; (3) the arithmetic means of the separate constituents will depend on the precise weighting factor ϕ_ν, but the combination of the arithmetic means will be independent of ϕ_ν.

Thus, if it happens that the separate opacities combine to give approximately "grey" absorption, there is a considerable advantage in calculating with the arithmetic means instead of the Rosseland means of the separate constituents. For in these circumstances we get the right result by combining the arithmetic means of the separate constituents as they stand, and the result is, moreover, independent of the weighting factor ϕ_ν used in the process.

The importance of this is in its application to the photospheric layers of the sun, and indeed of all stars which show an approximately black-body continuous spectrum. If the spectrum is black body, the absorption coefficient must be the same for all frequencies,* and we have the circumstances just contemplated. The measurements of Fabry and Buisson † and of H. H. Plaskett ‡ show that the sun's continuous spectrum is black body in the region 3000–6700 Å. Moreover, an approximately constant absorption coefficient is indicated by the observed intensity-distribution over the sun's disc in the separate wave-lengths. In a recent paper on absorption coefficients in the photospheric layers I showed that the opacity principally arises from photo-electric absorption.§ Since observation shows the sun's photospheric material to absorb as a "grey" body, it can only be the case that this arises from the superposition of regions of continuous photo-electric absorption, each sharply bounded on the long wave-length side and stretching indefinitely towards the higher frequencies. In each separate absorption region thus combined the absorption coefficient $k_\nu(\nu)$ varies roughly as ν^{-3}, and so decreases as ν increases; but this will, at least in part, be compensated by the fact that the higher frequencies can be absorbed by a greater number of atomic levels, the lower ones only by the atoms with the series electron in outer orbits. The smoothing-out of these discontinuously bounded tracts of absorption probably occurs in virtue of superpositions of such tracts from a number of different atoms.

In my paper, for reasons of convenience, I used the arithmetic mean

* If the coefficient was not constant, the existence of a temperature gradient in the photospheric layers would give rise to departures from a black-body spectrum: where the coefficient became smaller, we should see deeper and the emergent radiation would be more intense, and conversely.

† *Comptes Rendus*, **175**, 156, 1922.

‡ *Pub. Domin. Astrophys. Obs.*, **2**, 213, 1923.

§ *M.N.*, **85**, 768, 1925.

instead of the theoretically more correct Rosseland mean. But we see now that whatever the physical origin of the observed constancy of $k(\nu)$ for different ν for the sun's layers, it justifies the use of an arithmetic mean as equivalent to the Rosseland mean in determining the value of the absorption coefficient. Of course, one does not attempt to introduce actual proportions $a_1, \ldots a_n$; but the observed constancy gives some confidence that the error committed in using the arithmetic mean is relatively slight, so that the coefficients thus calculated do afford a true indication of the opacity of the mixture.

Whether a similar state of affairs holds in the interior of a star is more difficult to ascertain. Strictly speaking, the Rosseland mean absorption coefficient cannot be evaluated until we know the proportions in which the different constituents are present, unless the different constituents have absorption coefficients which all behave in the same way considered as functions of ν. Unless the latter holds, it is difficult to obtain any algebraically simple formula for the mean absorption coefficient, since the integrand involves the sum of the opacities in the denominator.

In his recent valuable paper * on the stellar absorption coefficient, Rosseland has effected the integration by increasing certain exponential factors until they are unity. This in effect reduces all the separate absorption coefficients to a single function of ν, thus making the integration possible, but as the result is then only an upper limit he has sacrificed some of the accuracy introduced by using his own mean value instead of an arithmetic mean. For iron, a pure substance, Rosseland obtained $k=10$ for Eddington's data for Capella. In my own paper † I obtained $k=6{\cdot}85$, or $k=8{\cdot}8$ on correcting for the empirical value of Kramer's "g'." I gave reasons for considering that the methods of summation and integration automatically accounted for part of Kramer's factor $g'=2{\cdot}3$, requiring only $g'=1{\cdot}3$. Had Rosseland used $g=1{\cdot}3$ he would have obtained $k=5{\cdot}7$. My value, $k=8{\cdot}8$, is larger than this. Rosseland's is, however, the theoretical upper limit for the pure element. There were other minor points of difference between our calculations besides our different ways of averaging.

The question whether Rosseland's calculations or mine more accurately represent the theoretical absorption coefficient for the physical conditions in a stellar interior turns largely on whether the general run of the total opacity, in the important region, is given by a formula of the type

$$k(\nu) \propto \nu^{-3} e^{h\nu/kT},$$

as used by Rosseland, or whether the different photo-electric limits of the different constituents in their different excited states so combine as to give an absorption coefficient approximately independent of ν. However improbable it may appear *a priori*, this seems to be so for the photospheric layers—say, at a temperature of 10^4 degrees. The question is whether the same holds at 10^7 degrees.

* *Astrophys. Journ.*, **61**, 424, 1925.
† *M.N.*, **85**, 750, 1925.

Appendix.

The first part of inequality (4) follows immediately from the well-known Schwarz's Inequality for integrals. The second part I have not seen explicitly stated. To prove it, it is clearly sufficient to prove that

$$\frac{1}{\int\frac{dx}{f(x)+g(x)}}\geqslant\frac{1}{\int\frac{dx}{f(x)}}+\frac{1}{\int\frac{dx}{g(x)}},$$

the integrals being taken through some given range. Put

$$1/f(x)=\mathrm{F}(x),\quad 1/g(x)=\mathrm{G}(x).$$

Then we have to prove that

$$\frac{1}{\int\frac{\mathrm{F}(x)\mathrm{G}(x)}{\mathrm{F}(x)+\mathrm{G}(x)}dx}\geqslant\frac{1}{\int\mathrm{F}(x)dx}+\frac{1}{\int\mathrm{G}(x)dx},$$

i.e. if we suppose that $\int\mathrm{F}(x)dx$ and $\int\mathrm{G}(x)dx$ are both positive, that

$$\Big(\int\mathrm{F}(x)dx\Big)\Big(\int\mathrm{G}(x)dx\Big)\geqslant\Big(\int\frac{\mathrm{F}(x)\mathrm{G}(x)}{\mathrm{F}(x)+\mathrm{G}(x)}dx\Big)\Big(\int[\mathrm{F}(x)+\mathrm{G}(x)]dx\Big)\qquad(5)$$

It is clearly sufficient to prove that

$$(\Sigma\mathrm{F}_r\triangle_r)(\Sigma\mathrm{G}_r\triangle_r)\geqslant\Big(\Sigma\frac{\mathrm{F}_r\mathrm{G}_r}{\mathrm{F}_r+\mathrm{G}_r}\triangle_r\Big)(\Sigma(\mathrm{F}_r+\mathrm{G}_r)\triangle_r),$$

i.e. taking the coefficient of $\triangle_r\triangle_s$, that

$$\mathrm{F}_r\mathrm{G}_s+\mathrm{F}_s\mathrm{G}_r\geqslant\frac{\mathrm{F}_r\mathrm{G}_r(\mathrm{F}_s+\mathrm{G}_s)}{\mathrm{F}_r+\mathrm{G}_r}+\frac{\mathrm{F}_s\mathrm{G}_s(\mathrm{F}_r+\mathrm{G}_s)}{\mathrm{F}_s+\mathrm{G}_s}.$$

This reduces to

$$(\mathrm{F}_r\mathrm{G}_s-\mathrm{F}_s\mathrm{G}_r)^2\geqslant 0,$$

provided F+G is always positive. The inequality therefore holds unless F/G=constant, in which case it reduces to an equality.

Applying Schwarz's Inequality once again, we have from (5)

$$\Big(\int\mathrm{F}(x)dx\Big)\Big(\int\mathrm{G}(x)dx\Big)\geqslant\Big(\int\frac{\mathrm{F}(x)\mathrm{G}(x)}{\mathrm{F}(x)+\mathrm{G}(x)}dx\Big)\Big(\int[\mathrm{F}(x)+\mathrm{G}(x)]dx\Big)\geqslant\Big(\int\sqrt{\mathrm{F}(x)\mathrm{G}(x)}dx\Big)^2\qquad(6$$

The inequality we have established is thus more delicate than Schwarz's, which consists of the first and third terms in (6).

On the Distribution of the Different Spectral Types along the Milky Way.
By Dr. E. A. Kreiken.

(*Communicated by the Secretaries.*)

1. In the *Monthly Notices* of March 1925, pp. 499–507, the curves of equal density in the Milky Way derived for distances between about 250 to 5000 parsecs from the numbers N_m * in the different regions in the Selected Areas, were given. From these curves it appears that our stellar universe is chiefly elongated in the directions $\lambda=270°$ and $\lambda=60°$. A scarcity of stars was found for λ about 0° and $\lambda=120°$.

In the present paper I have tried to ascertain whether these conclusions also hold good for the brighter stars of different spectral types.

From the Henry Draper Catalogue we find the numbers of B, A, F, G, K, and M stars brighter than magnitude 6·5; 7·5 and 8·5 in regions along the Milky Way 30° wide and with galactic latitude from +30° to +10°; +10° to −10°, and −10° to −30°. The results for these three zones are given in Tables I., II., and III. With the data from these tables I drew the figures 1, 2, and 3. The differences Δ (log N_m region −log N_m 120°–150°) have been plotted against the galactic longitudes. In fig. 2 the last curve represents the distribution of stars of all spectra taken together. It holds for all spectral types together, magnitude limit $m=15{\cdot}0$, and was taken from the *M.N.* (*loc. cit.*, p. 500, dotted line).

First we will consider the results for the central region with $|\beta|<10°$.

The distribution curves of the B, A, K, and M stars are very similar to that found for all stars together. The two maxima and minima are clearly defined.

The curves for the F and G stars are of a quite different character. They indicate that stars belonging to these two spectral types are distributed regularly over all longitudes.

For the K and M stars the same dependency on the galactic longitude is clear from figs. 1 and 3, while it disappears in the curves of the other spectral types.

I have tried to derive from our material equal-density curves for the smaller distances. As the range of magnitude is very small, it is not possible to find the coefficients of the apparent luminosity curve. Therefore we cannot use Schwarzschild equations for computing the density-function.

Consequently the solution which I give is only a first approximation.

Suppose

$$\phi(M)=e^{p+qM-rM^2} \quad . \quad . \quad . \quad . \quad (1)$$

is the absolute luminosity-function,

$$D(\rho)=e^{h+k\rho-l\rho^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the density-function;

* For the meaning of the symbols, cf. *M.N.*, March 1925, p. 499.

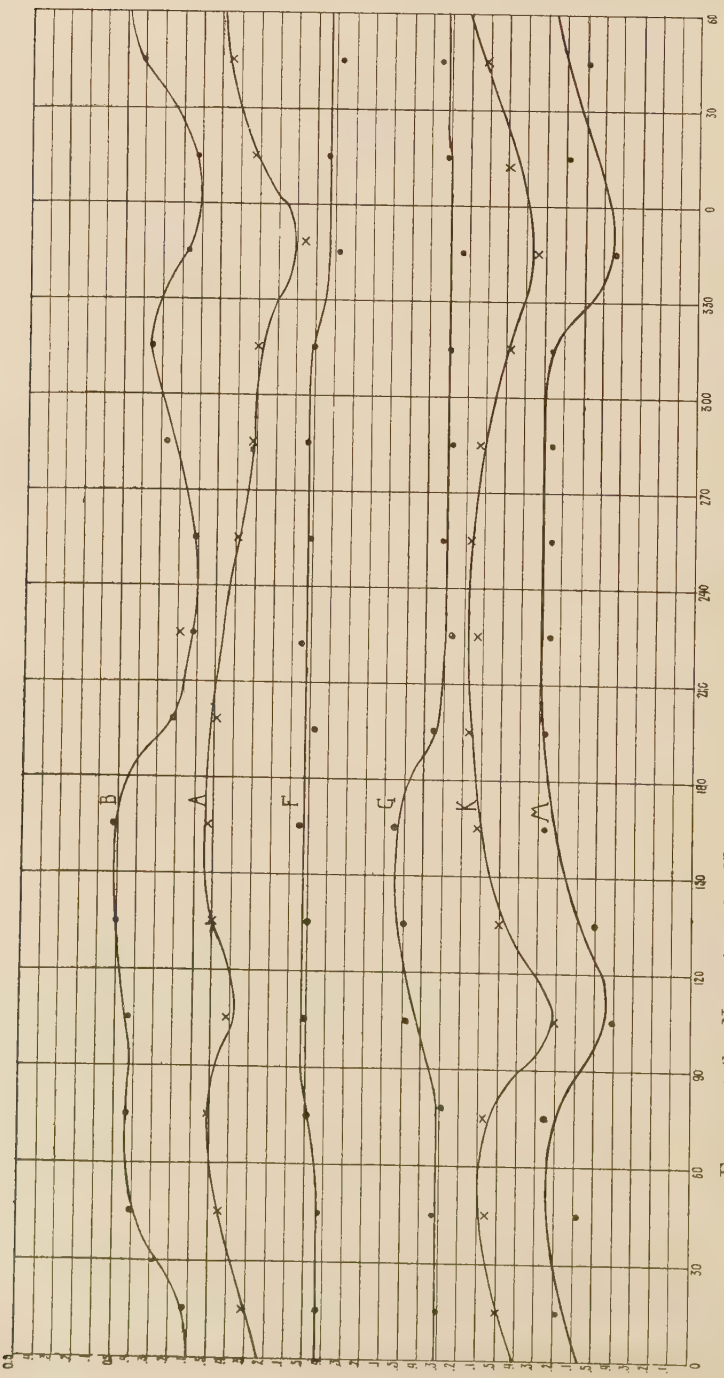


Fig. 1.—Δ (log N_m region −log N 120°–150°) for different longitudes and for latitude +30° to +10°.
Magnitude limit for all spectral types = 8·5.

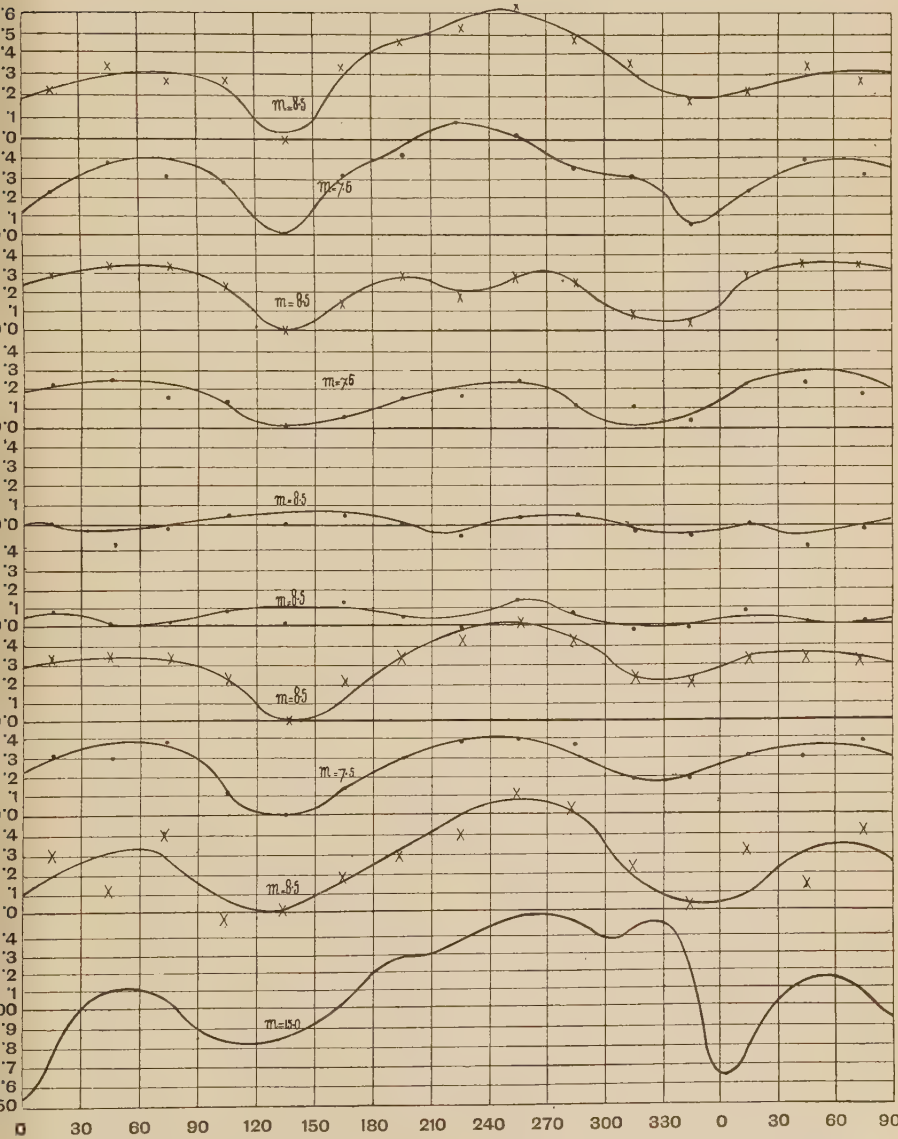


FIG. 2.—Δ (log N_m region −log N_m 120°–150°) for different longitudes and latitude +10° to −10° for the different spectral types. The curve for all stars together, $m=15{\cdot}0$, was taken from *M.N.*, **85**, 501.

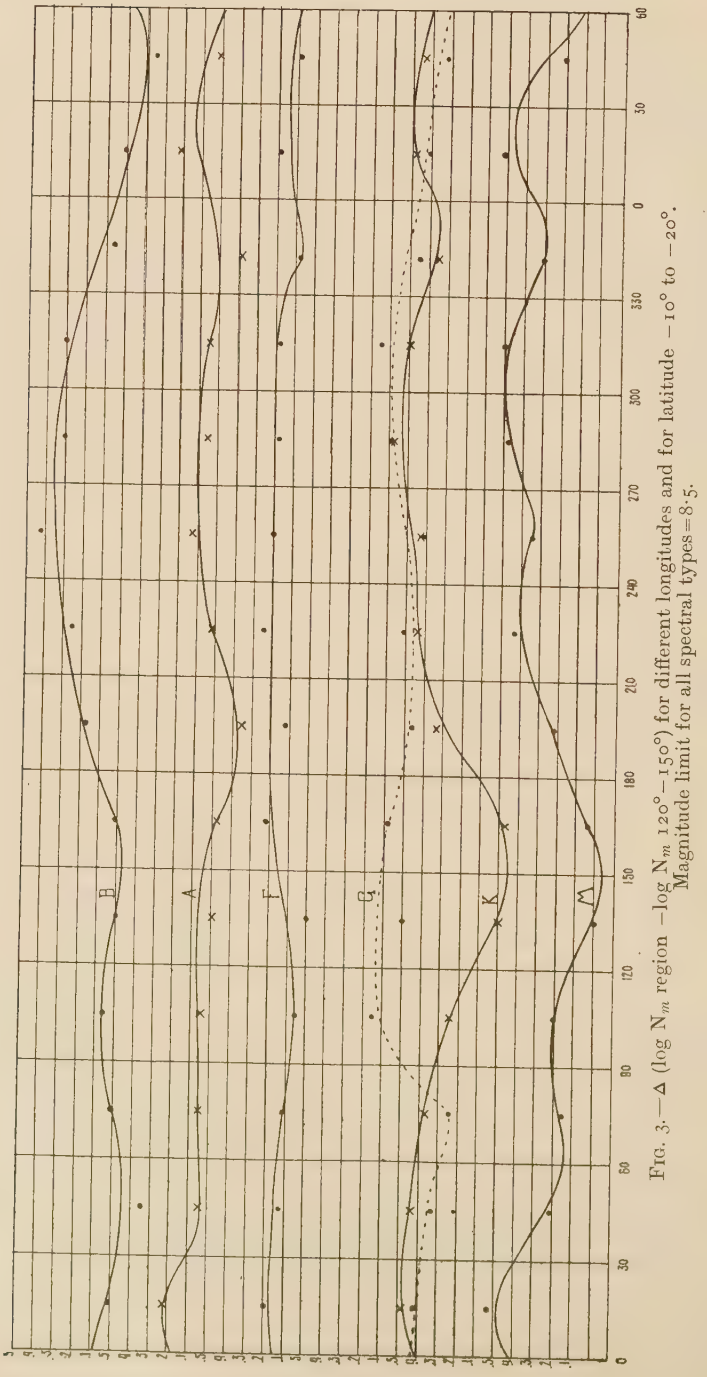


FIG. 3.—Δ (log N_m region −log N_m 120°–150°) for different longitudes and for latitude −10° to −20°.
Magnitude limit for all spectral types = 8·5.

TABLE I.

Log N_m, i.e. *Log Number of Stars per* 10,000 *Square Degrees for different Spectral Types and different Galactic Longitudes,* β *from* $+30°$ *to* $+10°$.

	m. \ λ.	0.	30°.	60°.	90°.	120°.	150°.	180°.	210°.	240°.	270°.	300°.	330°.	360°
B	6·5	1·176	1·699	1·623	1·322	1·398	1·255	1·146	1·279	1·415	1·820	1·898	1·568	
	7·5	1·653	2·072	2·000	1·851	1·944	1·833	1·633	1·716	1·690	2·017	2·127	1·857	
	8·5	1·959	2·243	2·262	2·260	2·334	2·369	2·045	1·954	1·940	2·114	2·185	2·013	
A	6·5	1·778	1·924	1·748	1·881	1·708	1·690	1·820	1·664	1·740	1·623	1·580	1·653	
	7·5	2·137	2·301	2·318	2·248	2·201	2·182	2·253	2·190	2·230	2·190	2·127	2·013	
	8·5	2·605	2·743	2·834	2·699	2·791	2·828	2·777	2·980	2·673	2·603	2·600	2·342	
F	6·5	1·176	1·431	1·505	1·415	1·447	1·602	1·322	1·342	1·204	1·477	1·279	1·462	
	7·5	1·959	1·982	2·041	2·013	1·908	2·000	1·944	2·057	1·973	2·083	2·017	1·944	
	8·5	2·484	2·468	2·549	2·563	2·550	2·601	2·525	2·602	2·558	2·584	2·559	2·413	
G	6·5	1·322	1·322	1·342	1·322	1·301	1·322	1·462	0·699	1·362	1·176	1·279	1·146	
	7·5	1·806	1·785	1·792	1·970	2·009	2·013	1·881	1·663	1·869	1·940	1·756	1·748	
	8·5	2·375	2·398	2·354	2·563	2·565	2·617	2·410	2·328	2·382	2·336	2·360	2·286	
K	6·5	1·740	1·845	1·785	1·724	1·778	1·881	1·672	1·820	1·681	1·959	1·914	1·591	
	7·5	2·246	2·364	2·324	2·124	2·297	2·288	2·272	2·310	2·346	2·281	2·262	2·083	
	8·5	2·735	2·810	2·822	2·449	2·742	2·862	2·918	2·874	2·914	2·874	2·723	2·573	
M	7·5	1·380	1·447	1·477	1·255	1·114	1·279	1·146	1·204	1·362	1·230	1·204	1·079	
	8·5	1·748	1·653	1·820	1·462	1·556	1·826	1·833	1·820	1·806	1·806	1·820	1·491	

Table II.

Log N_m, i.e. *Log Number of Stars per* 10,000 *Square Degrees for different Spectral Types and Galactic Longitudes,* β *from* $+10°$ *to* $-10°$.

	m. \ λ.	0°.	30°.	60°.	90°.	120°.	150°.	180°.	210°.	240°.	270°.	300°.	330°.	360°.
	6·5	1·908	1·987	1·875	1·964	1·778	2·062	2·086	2·250	2·130	1·996	1·806	1·613	
B	7·5	2·276	2·431	2·352	2·340	2·045	2·362	2·447	2·619	2·582	2·408	2·346	2·097	
	8·5	2·604	2·728	2·665	2·670	2·393	2·732	2·856	2·936	3·017	2·876	2·762	2·584	
	6·5	1·908	1·908	1·799	1·826	1·708	1·851	1·909	1·756	1·869	1·863	1·792	1·833	
A	7·5	2·405	2·410	2·352	2·326	2·176	2·233	2·328	2·334	2·427	2·297	2·283	2·233	
	8·5	2·972	3·009	3·009	2·895	2·675	2·823	2·965	2·846	2·964	2·827	2·768	2·737	
	6·5	1·531	1·204	1·322	1·591	1·568	1·255	1·519	1·568	1·531	1·591	1·462	1·531	
F	7·5	2·086	1·892	1·996	2·130	2·053	2·000	1·959	2·041	2·017	2·100	2·029	1·982	
	8·5	2·573	2·477	2·545	2·626	2·573	2·636	2·584	2·513	2·642	2·649	2·553	2·511	
	6·5	1·431	0·699	1·322	1·301	1·362	1·398	1·255	1·505	1·279	1·322	1·146	1·415	
G	7·5	1·940	1·724	1·968	1·954	1·982	1·919	1·771	1·964	1·881	1·919	1·886	1·908	
	8·5	2·476	2·400	2·407	2·486	2·396	2·537	2·459	2·391	2·547	2·490	2·371	2·386	
	6·5	1·778	1·908	1·954	1·681	1·568	1·778	1·908	1·863	1·968	1·924	1·740	1·756	
K	7·5	2·391	2·386	2·441	2·212	2·086	2·238	2·386	2·467	2·470	2·459	2·281	2·283	
	8·5	2·873	2·877	2·875	2·750	2·543	2·757	2·893	2·965	3·064	2·976	2·787	2·766	
M	7·5	1·447	1·230	1·653	1·000	..	1·301	1·362	1·505	1·690	1·531	1·230	1·176	
	8·5	1·756	1·591	1·892	1·380	1·447	1·643	1·740	1·833	2·057	2·000	1·690	1·491	

Table III.

Log N_m, i.e. *Log Numbers of Stars per* 10,000 *Square Degrees for different Spectral Types and Longitudes.*

	m. \ λ.	0°.	30°.	60°.	90°.	120°.	150°.	180°.	210°.	240°.	270°.	300°.	330°.	360°.
	6·5	1·362	1·415	1·656	1·479	1·949	1·792	1·940	1·919	1·930	1·693	1·663	1·623	
B	7·5	2·037	1·799	1·966	2·002	2·104	2·053	2·143	2·230	2·358	2·259	2·223	1·914	
	8·5	2·223	2·053	2·212	2·271	2·205	2·223	2·391	2·458	2·642	2·621	2·513	2·265	
	6·5	1·863	1·531	1·673	1·734	1·663	2·053	1·633	1·544	1·808	1·737	1·792	1·591	
A	7·5	2·330	2·143	2·217	2·206	2·137	2·262	2·097	2·225	2·336	2·314	2·286	2·124	
	8·5	2·889	2·718	2·731	2·738	2·671	2·639	2·531	2·700	2·796	2·808	2·731	2·560	
	6·5	1·431	1·672	1·513	1·718	1·146	1·934	1·602	1·580	1·548	1·200	1·176	1·447	
F	7·5	1·973	2·025	2·050	2·154	1·799	2·338	2·017	2·090	2·012	1·978	2·053	1·987	
	8·5	2·619	2·554	2·539	2·667	2·415	2·639	2·548	2·662	2·616	2·600	2·597	2·507	
	6·5	1·204	1·079	1·036	1·558	1·342	1·301	1·279	1·591	1·247	1·149	1·362	1·146	
G	7·5	1·740	1·580	1·735	2·059	1·875	2·004	1·934	2·064	1·774	1·790	2·076	1·863	
	8·5	2·384	2·301	2·212	2·633	2·471	2·554	2·431	2·479	2·391	2·560	2·623	2·425	
	6·5	1·845	1·556	1·777	1·834	1·491	1·740	1·869	1·833	1·930	1·945	1·940	1·771	
K	7·5	2·377	2·238	2·291	2·237	1·949	1·987	2·286	2·371	2·364	2·395	2·461	2·279	
	8·5	2·930	2·881	2·802	2·696	2·441	2·410	2·775	2·878	2·865	3·011	2·945	2·800	
M	7·5	1·544	1·000	1·337	1·257	..	1·079	1·279	1·398	1·247	1·422	1·362	1·079	
	8·5	1·935	1·613	1·559	1·604	1·146	1·431	1·623	1·833	1·737	1·879	1·908	1·708	

then, according to Pannekoek,* stars of one and the same apparent magnitude will be distributed in space round a median distance

$$\rho_m = \frac{0{\cdot}6+k-q+2_7 m}{l+r} \quad . \quad . \quad . \quad . \quad (3)$$

according to the Gaussian error curve (4).

$$D_1(\rho) = e^{-(l+r)(\rho-\rho_m)^2} \quad . \quad . \quad . \quad (4)$$

ρ_m denotes the distance which contributes most to the stars of magnitude m. Inversely all stars with one and the same distance ρ will have an apparent luminosity curve

$$\phi_1(m) = \text{const.} \times e^{-h^2(m-m_0)^2} \quad . \quad . \quad . \quad (5)$$

where

$$m_0 = M_0+\rho \quad . \quad . \quad . \quad . \quad . \quad (6)$$

m_0 is the most frequent apparent magnitude and M_0 the top of the absolute luminosity curve.

From (3) and (4) we find which fraction of the stars with an apparent magnitude m have a distance between $\rho=\rho_0\pm\frac{1}{2}$, etc.; while 5 and 6 give the fraction of the stars with the same distance ρ_0 which have an apparent magnitude brighter than $m=8{\cdot}5$. These two fractions enable us to find the density at distance ρ_0. Thus I have found, for instance, that B stars with distance $\rho=10{\cdot}5$ have the fraction 0·825 brighter than $m=8{\cdot}5$.

On the other hand, the fractions of the stars with $m<6{\cdot}5$, with m between 6·5 and 7·5 and between 7·5 and 8·5, which have a distance between $\rho=10{\cdot}5-\frac{1}{2}$ and $10{\cdot}5+\frac{1}{2}$ are 0·272 ; 0·200 ; 0·112.

So the equation

$$0{\cdot}825\ S_{10{\cdot}5} = {\cdot}242\ N_{6{\cdot}5} + {\cdot}200\ A_7 + {\cdot}112\ A_8$$

gives the total number of stars between the distances $\rho=10{\cdot}5\pm\frac{1}{2}$.

(N_m total number of stars per square degree brighter than magnitude m ; A_m total number of stars per square degree with apparent magnitude between $m-\frac{1}{2}$ and $m+\frac{1}{2}$.)

In applying this method we are greatly handicapped by the fact that the coefficients of the absolute luminosity curve are only known for the B and A stars.

§ 2. Equal density curves from the B stars.

The coefficients of the absolute luminosity curve were taken from Kapteyn's investigation (*Astrophysical Journal*, 7, 107, 1918).

h was taken as 0·44, *i.e.* the value found from the B_0–B_5 stars, and which is best known. For M_0, I adopted the value $-3{\cdot}5$, which is the mean of the values for the B_0–B_5 and B_5–B_9 stars.

* Dr. A. Pannekoek, *Publ. of Astr. Inst. of Amsterdam*, No. 1, p. 4.

For k and l I used the values derived for $\beta=0°$ by Kapteyn and van Rhijn.*

These values k and l represent a mean value for all longitudes and spectral types. When a second solution is made it will be necessary to derive them for each longitude separately.

The distribution of the stars of apparent magnitude m is given in Table V.

The distribution of the stars brighter than $m=6{\cdot}5$ was derived by computing it separately for $m=6{\cdot}0$; $5{\cdot}0$; $4{\cdot}0$, and $3{\cdot}0$.

Considering to what extent each magnitude contributes to the total number $N_{6\cdot5}$, we find the actual distribution for $m=6{\cdot}5$.

The fractions of stars with distance ρ for $m<6{\cdot}5$ which have an apparent magnitude $<8{\cdot}5$ are given in Table IV.

From the fractions in Tables IV. and V. we find the set of equations (7):

$$\left.\begin{aligned} {\cdot}940S_{9\cdot5} &= {\cdot}315N_{6\cdot5}+{\cdot}089A_7+{\cdot}035A_8 \\ {\cdot}825S_{10\cdot5} &= {\cdot}242N_{6\cdot5}+{\cdot}200A_7+{\cdot}112A_8 \\ {\cdot}622S_{11\cdot5} &= {\cdot}190N_{6\cdot5}+{\cdot}273A_7+{\cdot}224A_8 \\ {\cdot}378S_{12\cdot5} &= {\cdot}098N_{6\cdot5}+{\cdot}234A_7+{\cdot}276A_8 \end{aligned}\right\} \quad . \quad . \quad (7)$$

(S_ρ total number of stars between the limits $\rho-\frac{1}{2}$ and $\rho+\frac{1}{2}$.)

$S_{9\cdot5}$ is chiefly based on the value $N_{6\cdot5}$ and has only a small weight. Of the number $S_{12\cdot5}$ only about $\frac{1}{3}$ are observed.

TABLE IV.

Fractions of Stars with the same Distance ρ which are brighter than Magnitude 8·5

	Fractions.	
ρ.	B Stars.	A Stars.
6·5	1·000	1·000
7·5	1·000	·998
8·5	·985	·968
9·5	·940	·751
10·5	·825	·325
11·5	·622	·057
12·5	·378	·012

TABLE V.

Distribution of the Stars with Magnitude m for different values of ρ.

	Fractions.												
	B Stars.						A Stars.						
m \ ρ	$-\infty$.	9.	10.	11.	12.	13.	$-\infty$.	6.	7.	8.	9.	10.	11.	12.
6	·214	·315	·242	·190	·098		·101	·188	·311	·281	·097	·012	·000	
7	·030	·089	·200	·273	·234		·007	·090	·347	·398	·142	·016	·000	
8	·007	·035	·112	·224	·276		·000	·000	·002	·118	·369	·382	·118	

* J. C. Kapteyn and P. J. van Rhijn, *Astrophysical Journal*, **52**, 289.

TABLE VI.

Relative Density of B *and* A *Stars at Distance* ρ *for different Galactic Longitudes.*

	log D(ρ).							
	B Stars.				A Stars.			
λ \ ρ=	9·5.	10·5.	11·5.	12·5.	7·5.	8·5.	9·5.	10·5.
0°	9·905	9·534	9·239	8·830	0·595	0·335	0·061	9·791
30°	0·015	9·661	9·367	8·956	·604	·416	·201	9·943
60°	9·933	9·590	9·309	8·899	·552	·400	·213	9·954
90°	9·968	9·600	9·302	8·896	·517	·256	9·995	9·715
120°	9·729	9·317	9·015	8·610	·376	·062	9·749	9·467
150°	0·030	9·653	9·360	8·958	·423	·170	9·917	9·651
180°	0·109	9·763	9·488	9·096	·517	·300	0·068	9·808
210°	0·243	9·869	9·570	9·161	·529	·235	9·925	9·645
240°	0·216	9·912	9·654	9·268	·619	·340	0·049	9·773
270°	0·072	9·763	9·510	9·129	·486	·197	9·909	9·633
300°	9·942	9·657	9·404	9·016	·473	·194	9·838	9·553
330°	9·757	9·462	9·221	8·844	·423	·117	9·810	9·533
360°								

The final values of the density in Table VI. were derived from equations (7), the volumes of the consecutive spheres also having been taken into account. Unit of density is the mean value for the distance $\rho=9{\cdot}5$. From Table VI. the lines of equal density in VII. were found by interpolation.

§ 3. Lines of equal density from the A stars.

The same proceeding as for the B stars was followed throughout.

h was taken $=0{\cdot}80$ and $M_0=-1{\cdot}6$ (*cf.* J. C. Kapteyn, *loc. cit.*).

Tables IV. and V. contain the data for the distribution over ρ and m.

Now we find the set of equations (8).

$$\left.\begin{aligned} S_{6\cdot5} &= {\cdot}188N_6+{\cdot}020A_7 \\ {\cdot}998S_{7\cdot5} &= {\cdot}311N_6+{\cdot}347A_7+{\cdot}002A_8 \\ {\cdot}968S_{8\cdot5} &= {\cdot}281N_6+{\cdot}398A_7+{\cdot}118A_8 \\ {\cdot}751S_{9\cdot5} &= {\cdot}097N_6+{\cdot}142A_7+{\cdot}369A_8 \\ {\cdot}325S_{10\cdot5} &= {\cdot}012N_6+{\cdot}016A_7+{\cdot}382A_8 \end{aligned}\right\} \quad . \quad . \quad (8)$$

$S_{7\cdot5}$, $S_{8\cdot5}$, and $S_{9\cdot5}$ are well determined. $S_{6\cdot5}$ and $S_{10\cdot5}$ have but small weight. The relative densities found from (8) have been entered in Table VI. As unit of density we have taken again the mean value for $\rho=9{\cdot}5$. From Table VI. the equal density curves given in Table VII. were derived.

From the values in Table VII. fig. 4 was drawn.

TABLE VII.

Distances for Equal Densities.

λ \ log D(ρ).	B Stars. ρ in Parsecs. 9·9.	9·7.	9·5.	9·3.	9·1.	8·9.	A Stars. ρ in Parsecs. 0·4.	0·2.	0·0.	9·8.	6·9.
0°	80	100	131	182	230	289	44	63	87	126	(182)
30	87	120	166	219	272	(347)	52	80	115	(170)	..
60	80	105	145	200	251	316	50	80	115	(170)	..
90	83	115	145	200	251	316	42	55	80	110	(151)
120	..	80	100	126	166	230	(30)	42	55	72	96
150	91	120	159	219	263	(330)	34	48	69	96	138
180	105	126	191	240	316	..	40	60	93	126	(182)
210	120	159	219	263	331	..	42	52	72	102	138
240	125	182	240	300	..	..	46	64	87	120	(176)
270	100	138	200	263	331	..	36	50	69	91	132
300	83	115	166	230	276	..	36	50	66	84	115
330	..	76	115	175	219	289	33	44	60	80	115
360											

The values in brackets have been extrapolated.

The dotted line is the equal density curve log D(ρ)=0 for A stars, the full line the curve log D(ρ)=9·5 for B stars.

I also give in fig. 4 the equal density curve log D(ρ)=9·1, which has been taken from the *M.N.* (*loc. cit.*, p. 505), though on a scale forty times smaller. In the curve of the A stars only the elongation at λ=60° is plainly visible, the one at λ 270° is only faintly indicated. In the curve of the B stars, however, the latter is predominant.

The whole character of that curve is the same as that for all stars together at greater distances.

The main results of the present paper are summarised in the following :—

1. The numbers of B, A, K, and M stars depend systematically on galactic longitude. For $|\beta| < 10$ the shape of the frequency curves is the same as that found for fainter stars, all spectral types taken together.

2. The equal density curves derived from the B and A stars show elongations of our stellar universe in the directions λ=60° and λ=270°. A lack of stars appears in the directions λ about 0° and about 120°.

These same irregularities were found in the curves of the fainter stars in the Selected Areas (*M.N.*, March 1925, p. 499).

3. The G and F stars are distributed regularly over all longitudes.

4. In Harvard Circulars 240 and 248, H. Shapley has examined

the distribution of the different spectral types. He counted the star numbers in selected fields. His results are in many respects analogous to those found here from the complete material.

The irregularities are ascribed by him to galactic star clouds and to obscuring nebulosities. From the fact that the F and G stars are

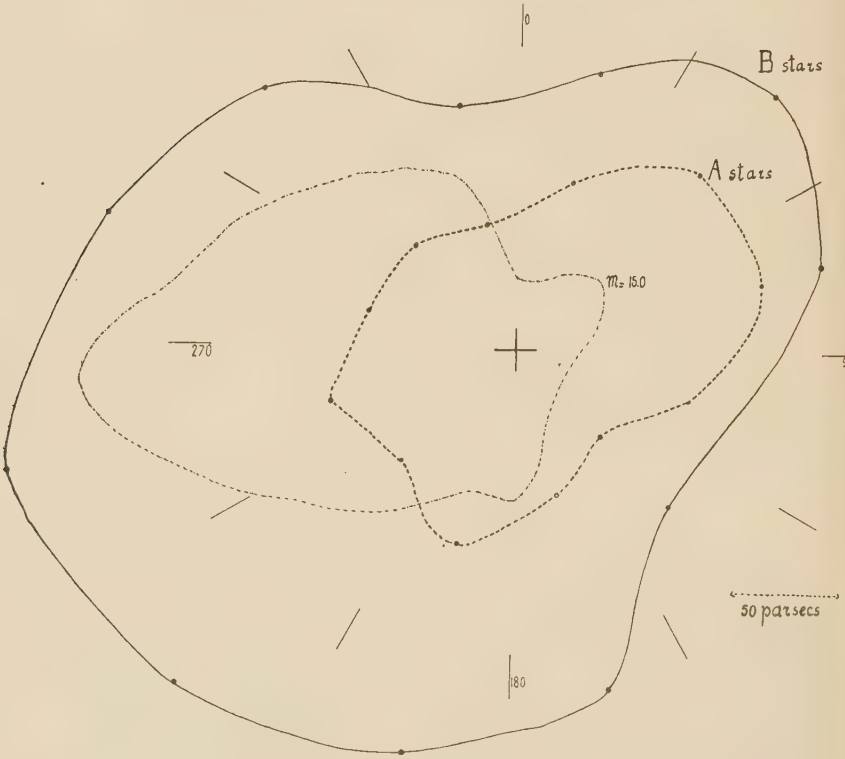


FIG. 4.—Curves of equal density for the B stars (log $D_{(\rho)}=9{\cdot}5$), for the A stars (log $D_{(\rho)}=0{\cdot}0$), and for all stars together $m=15{\cdot}0$ (log $D_{(\rho)}=9{\cdot}1$). The scale of the latter curve is forty times smaller than that of the two others. It was taken from *M.N.*, **85**, 505.

unaffected he concludes that the majority of the F and G stars are dwarfs.

5. From the curves of equal density published in *M.N.*, March 1925, and from those found in the present investigation, I conclude that the irregularities in the distribution curves of the B, A, K, and M stars are mainly due to the elongated shape of our stellar universe.

6. In the *Bulletin of the Astr. Institutes of the Netherlands*, **2**, No. 79, p. 59, the distribution of the stars in the Pleiades was examined. It appears that the cluster stars of magnitude 11·5 are widely scattered. The brighter and fainter cluster stars are almost entirely

confined to the immediate surroundings of the centre. The cluster stars of magnitude 11·5 are all of spectral class *f* and *g*, those brighter than *m*−11·5 are *b* or *a* stars, while the fainter ones are K and *m* stars.

If this same feature holds for the general distribution of the stars in the stellar universe, viz. that the B, A, K, and M stars are strongly concentrated while the F and G stars are widely scattered, it can easily be seen that the distribution curve of the F and G stars must be unaffected by the irregularities which appear in the curves of the other spectral types.

Haarlem, Holland:
1925 *August.*

Preliminary Ephemeris of the Planet (433) *Eros for the Opposition of* 1930–31. By G. Witt.

(*Communicated by the Secretaries.*)

The places are for the mean equinox of 1925·0, and are referred to the centre of gravity of the earth-moon system.

Date. Greenwich Mean Noon.	α.	δ.	Parallax.	Mag.
1930.	h m	° ′	″	
Oct. 1	5 25·4	+44 2	12·2	10·5
2	29·0	44 14		
3	32·5	44 26		
4	36·0	44 38		
5	39·6	44 50	12·8	10·4
6	43·2	45 1		
7	46·8	45 12		
8	50·5	45 23		
9	54·2	45 33	13·4	10·3
10	5 57·9	45 43		
11	6 1·6	45 53		
12	5·4	46 2		
13	9·1	46 11	14·0	10·2
14	12·9	46 20		
15	16·7	46 28		
16	20·5	46 36		
17	24·4	46 44	14·6	10·1
18	28·2	46 51		
19	32·1	46 58		
20	35·9	47 4		
21	39·8	47 10	15·3	10·0
22	43·7	47 15		
23	6 47·7	+47 20		

Date. Greenwich Mean Noon.	α.	δ.	Parallax.	Mag.
1930.	h m	° ′	″	
Oct. 24	6 51·6	+47 25		
25	55·6	47 29	16·0	9·8
26	6 59·5	47 33		
27	7 3·5	47 36		
28	7·4	47 38		
29	11·4	47 40	16·8	9·7
30	15·4	47 42		
31	19·4	47 43		
Nov. 1	23·4	47 44		
2	27·3	47 44	17·6	9·6
3	31·3	47 44		
4	35·3	47 43		
5	39·3	47 41		
6	43·3	47 39	18·5	9·5
7	47·2	47 36		
8	51·1	47 33		
9	55·0	47 29		
10	7 58·9	47 25	19·4	9·3
11	8 2·8	47 20		
12	6·7	47 14		
13	10·6	47 8		
14	14·4	47 1	20·4	9·2
15	8 18·2	+46 54		

Date. Greenwich Mean Noon.	α.	δ.	Parallax.	Mag.
1930.	h m	° ′	″	
Nov. 16	8 22·0	+46 46		
17	25·8	46 37		
18	29·5	46 28	21·5	9·1
19	33·2	46 18		
20	36·9	46 7		
21	40·6	45 55		
22	44·2	45 43	22·7	9·0
23	47·8	45 31		
24	51·3	45 17		
25	54·8	45 3		
26	8 58·3	44 48	23·9	8·8
27	9 1·7	44 33		
28	5·1	44 16		
29	8·4	43 59		
30	11·6	43 42	25·2	8·7
Dec. 1	14·8	43 23		
2	18·0	43 4		
3	21·2	42 44		
4	24·3	42 23	26·6	8·6
5	27·3	42 1		
6	30·2	41 39		
7	33·1	41 16		
8	36·0	40 52	28·2	8·4
9	38·8	40 27		
10	41·5	40 1		
11	44·2	39 34		
12	46·8	39 7	29·8	8·3
13	49·3	38 38		
14	51·8	38 9		
15	54·2	37 39		
16	56·5	37 8	31·6	8·1
17	9 58·8	36 36		
18	10 1·0	36 3		
19	3·1	35 29		
20	5·2	34 54	33·5	8·0
21	7·2	34 18		
22	9·2	33 41		
23	11·1	33 3		
24	12·8	32 23	35·4	7·9
25	14·5	31 43		
26	16·1	31 2		
27	10 17·6	+30 20		

Date. Greenwich Mean Noon.	α.	δ.	Parallax.	Mag.
1930.	h m	° ′	″	
Dec. 28	10 19·1	+29 36	37·5	7·7
29	20·5	28 52		
30	21·8	28 6		
31	23·1	27 19		
1931.				
Jan. 1	24·3	26 31	39·6	7·6
2	25·3	25 42		
3	26·3	24 51		
4	27·2	24 0		
5	28·0	23 7	41·8	7·5
6	28·7	22 13		
7	29·4	21 18		
8	30·0	20 22		
9	30·5	19 25	43·9	7·4
10	30·9	18 27		
11	31·2	17 28		
12	31·4	16 27		
13	31·6	15 26	45·9	7·3
14	31·7	14 24		
15	31·7	13 21		
16	31·6	12 17		
17	31·4	11 12	47·6	7·2
18	31·1	10 7		
19	30·8	9 1		
20	30·4	7 55		
21	29·9	6 48	49·0	7·1
22	29·3	5 41		
23	28·7	4 34		
24	28·0	3 26		
25	27·2	2 19	49·9	7·1
26	26·3	1 12		
27	25·4	+ 0 5		
28	24·4	− 1 2		
29	23·4	2 8	50·3	7·1
30	22·3	3 13		
31	21·1	4 17		
Feb. 1	19·9	5 21		
2	18·6	6 24	50·2	7·1
3	17·3	7 25		
4	16·0	8 26		
5	14·6	9 25		
6	10 13·2	−10 22	49·5	7·1

Date. Greenwich Mean Noon.		α.	δ.	Parallax.	Mag.
1931.		h m	° ′	″	
Feb.	7	10 11·7	−11 18		
	8	10·2	12 13		
	9	8·7	13 6		
	10	7·2	13 57	48·3	7·2
	11	5·7	14 46		
	12	4·2	15 34		
	13	2·7	16 20		
	14	10 1·2	17 4	46·7	7·3
	15	9 59·7	17 46		
	16	58·2	18 26		
♂	17	56·7	19 4		
	18	55·2	19 41	44·9	7·4
	19	53·7	20 15		
	20	52·3	20 48		
	21	50·9	21 19		
	22	49·6	21 48	42·9	7·5
	23	48·3	22 15		
	24	47·0	22 41		
	25	45·8	23 5		
	26	44·7	23 27	40·9	7·6
	27	43·6	23 48		
	28	42·5	24 7		
Mar.	1	41·5	24 25		
	2	40·6	24 41	38·8	7·7
	3	39·7	24 56		
	4	38·9	25 9		
	5	38·2	25 22		
	6	37·5	25 33	36·7	7·9
	7	36·9	25 42		
	8	36·3	25 51		
	9	35·8	25 59		
	10	35·4	26 5	34·8	8·0
	11	35·1	26 11		
	12	34·8	26 15		
	13	34·6	26 19		
	14	34·4	26 22	32·9	8·1
	15	34·3	26 24		
	16	34·3	26 25		
	17	34·4	26 26		
	18	34·5	26 26	31·1	8·3
	19	34·7	26 25		
	20	34·9	26 24		
	21	35·2	26 22		
	22	9 35·5	−26 20	29·4	8·4

Date. Greenwich Mean Noon.		α.	δ.	Parallax.	Mag.
1931.		h m	° ′	″	
Mar.	23	9 35·9	−26 17		
	24	36·4	26 13		
	25	36·9	26 9		
	26	37·5	26 5	27·8	8·5
	27	38·2	26 1		
	28	38·9	25 56		
	29	39·7	25 51		
	30	40·5	25 45	26·3	8·7
	31	41·3	25 40		
Apr.	1	42·2	25 34		
	2	43·2	25 28		
	3	44·2	25 21	24·9	8·8
	4	45·3	25 15		
	5	46·4	25 8		
	6	47·5	25 1		
	7	48·7	24 54	23·6	9·0
	8	49·9	24 47		
	9	51·2	24 40		
	10	52·5	24 33		
	11	53·8	24 26	22·4	9·1
	12	55·2	24 19		
	13	56·6	24 11		
	14	58·1	24 4		
	15	9 59·6	23 57	21·2	9·2
	16	10 1·1	23 50		
	17	2·7	23 42		
	18	4·3	23 35		
	19	5·9	23 28	20·1	9·4
	20	7·5	23 21		
	21	9·2	23 14		
	22	10·9	23 7		
	23	12·6	23 0	19·1	9·5
	24	14·3	22 53		
	25	16·1	22 47		
	26	17·9	22 40		
	27	19·7	22 34	18·1	9·6
	28	21·5	22 27		
	29	23·4	22 21		
	30	25·3	22 15		
May	1	27·2	22 9	17·2	9·8
	2	29·1	22 3		
	3	31·0	21 58		
	4	33·0	21 52		
	5	10 35·0	−21 47	16·4	9·9

Berlin :
1925 *May.*

Position and Motion of the Equinox. By R. T. Cullen.

(*Communicated by the Astronomer Royal.*)

1. The determination of the position of the equinox and of its motion is one of the most important of the problems in meridian astronomy. Newcomb undertook a very complete investigation in 1872 in forming his system of standard right ascensions. His results depended upon observations made at various observatories, beginning with Bradley's observations about 1750 and concluding with the observations around 1870. Later he re-examined the question very thoroughly in his *Astronomical Constants*, subsequently adopting the same values for the position and movement of the equinox in his *Fundamental Catalogue for* 1900. Boss also examined the question for his *Preliminary General Catalogue*, and adopted Newcomb's original system. Auwers adopted nearly the same system as regards equinox. This may be considered as the best result obtainable from observations up to 1900.

More recently, improvements in the methods of observation have led to more concordant determinations, which, however, differ from the values obtained by Newcomb. In particular, Eichelberger, from a discussion of Washington and Cape observations, has determined anew the position of the equinox for 1910; and by comparison with Boss's mean equinox for 1840 has re-determined the motion. These differ considerably from Newcomb's values.

2. The Greenwich observations made with the Transit Circle, extending over the period 1850 to 1925, supply data which now afford a good opportunity for investigation of this subject. These successive determinations, each the mean of a group of years, shown in the following table and given in the Introduction to the *First* 1925 *Catalogue*, represent the corrections to Newcomb's Equinox:—

Year.	Correction.	Year.	Correction.
	s		s
1851	·000	1891	+·002
1857	−·019	1901	−·014
1864	−·032	1911	−·045
1872	−·029	1919	−·073
1882	−·035		

The most recent determination, that of 1922–1924, gives $-^{s}·059$ to the adopted equinox or $-^{s}·064$ to Newcomb's equinox, in close agreement with other values,

s	
−·064	1907–1911
−·053	1912–1916

obtained from Sun observations at the Cape and

s	
−·053	1911–1918

from those at Washington.

The period since 1891 (see above table) gives corrections that compel further examination. The Greenwich clock star systems in use before and during the 1890 Catalogue (1887–1896) were based upon the equinox determined from observations of the Sun, or agreed so closely with the adopted equinox as to require no revision. The period 1887–1896, however, produced a large correction, $+^{s}\cdot047$, to the adopted equinox. The years immediately following supported this, whereupon it was decided that from 1903 Newcomb's system, at that time recently published in his *Fundamental Catalogue for* 1900, and agreeing with the observed equinox, be adopted for the Greenwich clock star system. Since then, the most noticeable feature has been the rapid divergence, amounting to about $^{s}\cdot06$ at the present time; and this paper deals with this point in particular.

Before proceeding, it may be well to state that the equinox correction was derived at Greenwich in the past from discussion of deduced values of apparent corrections to ecliptic polar distances; but, for our purpose, the method by Newcomb, in which apparent corrections to R.A. and Dec. are dealt with separately, given in detail in his *Compendium*, pp. 325–330, has been employed.

The position of the equinox depends upon two elements to be obtained from observation :—

1. The determination of the correction to the Sun's absolute R.A. from the observations of its declination.
2. That of the differences between the R.A.'s of the stars and of the Sun.

The first term, denoted by $\Delta\alpha\odot=\Delta\lambda$, is derived from the mean monthly differences "Observed–Tabular" in declination. These give equations of the form $\sin\epsilon\Delta\lambda\cos\alpha+\Delta\epsilon\sin\alpha+\Delta_0=\Delta\delta$. The coefficient of $\cos\alpha$ divided by $\sin\epsilon$ gives the quantity $\Delta\lambda$.

The second term, $\Delta'\alpha\odot$, is the mean value of the residuals in R.A. in the sense "Observed–Tabular." Then the equinox correction, or correction to the assumed R.A.'s of the clock star system, is given by

$$\Delta\lambda-\Delta'\alpha\odot.$$

We can now proceed in detail to the results obtained from the observations extending from 1897·0 to 1925·0. The year 1897 was chosen as a starting-point, because corrections for variation of latitude were first applied to the Greenwich declinations published in the 1900 Catalogue (1897–1905).

The observations 1897-1905 are separated into two groups, 1897–1900 and 1901–1905, because in 1901 Newcomb's *Tables of the Sun* were first introduced into the *Nautical Almanac*, superseding those of Leverrier.

The mean monthly differences "Observed–Tabular" of the Sun in the different groups are as follows :—

[Table

	R.A. (Unit ·001).					Dec.				
	1897–1900.*	1901–1905.	1906–1914.	1915–1921.	1922–1924.	1897–1900.*	1901–1905.	1906–1914.	1915–1921.	1922–1924.
	s	s	s	s	s	″	″	″	″	″
Jan.	−126	− 6	+52	+108	+109	−·36	−·07	+·27	+·22	+·07
Feb.	− 86	+16	+55	+ 96	+ 90	−·05	−·22	+·19	+·23	+·28
Mar.	− 72	+ 5	+41	+ 84	+ 89	−·44	−·24	−·02	+·10	+·19
Apr.	− 62	+ 5	+48	+ 77	+113	−·13	−·13	+·05	+·03	−·10
May	− 92	−14	+36	+ 88	+117	+·25	+·25	+·26	+·14	−·04
June	− 95	−23	+43	+ 91	+114	+·17	−·05	−·25	−·05	−·16
July	−106	+10	+44	+120	+120	+·51	+·23	+·13	−·13	−·11
Aug.	−114	+12	+59	+123	+128	+·64	+·18	+·17	−·05	−·19
Sept.	−129	+ 8	+74	+109	+120	+·56	−·21	−·01	−·28	−·57
Oct.	−160	+30	+52	+115	+118	+·56	−·09	−·11	−·30	−·82
Nov.	−140	+ 5	+78	+108	+122	−·07	+·05	−·12	−·16	−·09
Dec.	−122	+10	+47	+115	+109	−·02	+·12	+·17	+·10	−·07
Mean	−109†	+ 5	+52	+103	+112	+·14	−·02	+·06	−·01	−·13

The analysis of the monthly differences in declination give

1897–1900*	+″·135	+″·244 sin α	−″·395 cos α.
1901–1905	−·015	+·076 ,,	−·072 ,,
1906–1914	+·061	−·030 ,,	+·070 ,,
1915–1921	−·012	−·050 ,,	+·208 ,,
1922–1924	−·134	−·051 ,,	+·321 ,,

From which we find

		Δλ.	Δ′α⊙.	Correction to adopted Equinox. Δλ−Δ′α⊙.
		s	s	s
1897–1900*	−″·395 cosec ε	=−·067	−·055	−·012
1901–1905	−·072 ,,	=−·012	+·005	−·017
1906–1914	+·070 ,,	=+·012	+·052	−·040
1915–1921	+·208 ,,	=+·035	+·103	−·068
1922–1924	+·321 ,,	=+·053	+·112	−·059

These results depend on the observed declinations as reduced with Pulkowa refractions, R−D+″·60 sin ZD, and corrections for variation of latitude.

We shall now discuss each term independently, and the effect produced on each, resulting in amended corrections to the equinox comparable with determinations made previous to the present century.

It is important to investigate whether there are any systematic differences between these observations and those of last century, because

* Leverrier's Tables, 1897–1900.

† Or −ˢ·055 referred to Newcomb's equinox.

it is necessary that a uniform system and method of treating them should be the first consideration.

3. Taking the term $\Delta' \alpha \odot$ first, namely, the differences of the R.A.'s of the stars and of the Sun. It is obvious that any marked change of personality affecting the observations of the Sun as compared with the stars must enter with full effect into this term. Now during the period under review, 1897–1924, a radical alteration took place in the method of observing transits by the introduction of the moving wire at the beginning of 1915. The effect is seen in the following table giving the mean annual differences "Observed–Tabular" of the Sun immediately before and after the change of method :—

Observations of the Sun in R.A. (O–T).

Tapping Method.		Moving Wire.	
	s		s
1911	+·064	1915	+·093*
1912	+·070	1916	+·110
1913	+·060	1917	+·102
1914	+·061	1918	+·107

The means corresponding to each are

	s
1911–1914	+·064
1915–1918	+·104

The difference s·040 must be attributed to a large extent to a change of personality caused by the different method of observing. The same regular observers were more or less common to both periods. Further, the mean annual results in R.A. for the Sun since 1915 are as follows :—

	s		s
1915	+·093	1920	+·102
1916	·110	1921	·103
1917	·102	1922	·117
1918	·107	1923	·106
1919	·112	1924	·113

They are so consistent that it may be safely inferred that the difference between the years 1911–1914 and 1915 onwards is due to the personality of the observers during the earlier years. It cannot be asserted that this same personality held for all the Greenwich observations before 1915, but this is the best assumption that can be made in view of the number of observers.

Another small change that should be noted, entering into this term when comparison is made with Newcomb's equinox, is one of s·005, introduced by corrections for magnitude equation into the Greenwich clock star system since 1915.

4. We now turn to the term $\Delta \alpha \odot = \Delta \lambda$ derived from the mean monthly differences from observations of the Sun in declination. The

* Half-weight. April to December only.

observations in declination can be separated into two groups, the distinction being the application or otherwise of a correction due to variation of latitude.* The first series, 1850–1896, was reduced *without*, the second, 1897–1924, *with* corrections for this variation. It will be of interest to see what effect on the term this correction has during the period 1897–1924.

Grouping the latitude variation corrections (applicable to observed declinations) into monthly means, as is done for the Sun observations, we get—

Corrections for Variation of Latitude.

	1897–1900.	1901–05.	1906–14.	1915–21.	1922–24.
	″	″	″	″	″
Jan.	·00	·00	+·02	−·07	−·12
Feb.	−·05	−·04	+·01	−·10	−·16
Mar.	−·09	−·06	−·01	−·12	−·17
Apr.	−·10	−·05	·00	−·11	−·17
May	−·10	−·03	+·01	−·07	−·14
June	−·07	+·01	+·03	·00	−·11
July	−·03	+·06	+·06	+·08	−·05
Aug.	+·02	+·09	+·09	+·12	+·06
Sept.	+·07	+·11	+·08	+·12	+·09
Oct.	+·10	+·11	+·07	+·09	+·04
Nov.	+·10	+·10	+·05	+·03	−·02
Dec.	+·08	+·07	+·02	−·02	−·04

The above corrections for 1897–1914 were determined by the International Bureau; those from 1915 by observations made with the Cookson telescope at Greenwich.

Analysis gives

	″	″	″
1897–1900	−·006	−·059 sin α	−·089 cos α.
1901–1905	+·031	−·013 ,,	−·090 ,,
1906–1914	+·036	+·008 ,,	−·044 ,,
1915–1921	−·004	+·024 ,,	−·122 ,,
1922–1924	−·066	−·007 ,,	−·125 ,,

affecting the term $\Delta\lambda$ as follows :—

	$\Delta\lambda$.
	s
1897–1900	−·015
1901–1905	−·015
1906–1914	−·007
1915–1921	−·020
1922–1924	−·020

* After this paper was in proof it was found that the effect of latitude variation had been discussed by Mr. W. G. Thackeray in a paper published in the *Monthly Notices* for March 1898, **58**, 280.

These results, arising from the correction for variation of latitude, show the extent to which the annual term of the variation affects the equinox correction. The mean value for the period 1897–1924 indicates an annual term in the latitude variation of ″·09, the maxima occurring about March and September, and, in consequence, entering into the equinox determination with full effect.

5. Having dealt with each term independently, a summary of the results can now be made. Taking the declination term first :—

$\Delta\lambda$ *Term.*

Period.	As determined *with* Lat. Var. Corrn.	Correction due to Lat. Var.	As deduced *without* Lat. Var. Corrn.
	s	s	s
1897–1900	−·067	−·015	−·052
1901–1905	−·012	−·015	+·003
1906–1914	+·012	−·007	+·019
1915–1921	+·035	−·020	+·055
1922–1924	+·053	−·020	+·073

Next, the term depending on the R.A.'s :—

$\Delta'\alpha\odot$ *Term.*

Period.	Determined Corrn. to Newcomb's System.	Correction due to changed Method of Observing.	Correction to Newcomb's System as deduced to a uniform System of Observing.
	s	s	s
1897–1900	−·055	..	−·055
1901–1905	+·005	..	+·005
1906–1914	+·052	..	+·052
1915–1921	+·108	−·040	+·068
1922–1924	+·117	−·040	+·077

The equinox correction $\Delta\lambda - \Delta'\alpha\odot$ becomes

Period.	Correction to Newcomb's Equinox.	
	As determined.	Corrected.
	s	s
1897–1900	−·012	+·003
1901–1905	−·017	−·002
1906–1914	−·040	−·033
1915–1921	−·073	−·013
1922–1924	−·064	−·004

In conclusion, there appears to be little doubt that from recent determinations Newcomb's equinox is in error to the extent of about s·06 at the present time, agreeing well with other contemporary determinations. The accordance of these results in general is probably due to the now universal system of observing transits with the moving wire, which unquestionably reduces the range of personality within narrow limits.

That from the Greenwich results the recent apparent rapid divergence from Newcomb's equinox, as compared with earlier determinations, may be accounted for by the change of method in the observing, coupled with the effect produced by the annual term in the correction applied for variation of latitude.

As to the motion of the equinox, the Greenwich corrections to Newcomb's equinox, reduced to a system of reduction as uniform as it is possible to attain, over the period of 75 years with the same instrument, are as follows :—

	s		s
1851	·000	1891	+·002
1857	−·019	1901	·000
1864	−·032	1911	−·033
1872	−·029	1919	−·013
1882	−·035	1923	−·004

Attention may again be drawn to the fact that the results quoted for 1919 and 1923 are corrected by ^{s}·040 for personality as compared with the observations made immediately before the introduction of the moving wire (see § 3). Naturally, it cannot be inferred that this correction for change of personality is applicable to the whole series 1850–1911, but no doubt exists that a correction of this order must be applied to bring the results from 1915 onwards into sympathy with those immediately before.

From these figures it is impossible to say whether the motion of the equinox, as adopted by Newcomb, Boss, and Auwers, requires a positive or negative correction.

This does not support Eichelberger's recent determination of a correction of $-^{s}$·06 per century, but fair agreement is possible if the Greenwich results, as determined recently and uncorrected, are compared, without regard to uniformity of system, with earlier determinations.

Royal Observatory, Greenwich :
1925 *September* 11.

Mean Areas and Heliographic Latitudes of Sunspots in the Year 1924.

(*Communicated by the Astronomer Royal.*)

The results here given are in continuation of those in the *Monthly Notices*, **84**, 740, and are derived from the measurement of photographs taken at the Royal Observatories of Greenwich and the Cape, and in India at the Kodaikánal Observatory and at Dehra Dûn.

Table I. gives the mean daily area of umbræ, whole spots, and faculæ for each synodic rotation of the Sun included in the year 1924, whilst the mean values for each year as a whole from 1913 to 1924 are given in Table II.

TABLE I.

No. of Rotation.	Rotation commenced G.M.T. (Civil).	No. of Days on which Sun photographed.	Mean of Daily Areas. Projected.* Umbræ.	Whole Spots.	Faculæ.	Corrected for Foreshortening.† Umbræ.	Whole Spots.	Faculæ.
	1923. d.							
940	Dec. 25·41	28	1	6	179	1	6	218
	1924.							
941	Jan. 21·75	27	1	4	103	½	4	116
942	Feb. 18·09	27	26	146	77	20	107	90
943	Mar. 16·42	28	1	4	110	½	2	126
944	Apr. 12·71	27	43	266	145	28	178	183
945	May 9·95	27	90	573	457	68	445	532
946	June 6·16	27	61	387	433	51	323	514
947	July 3·36	28	93	496	804	66	367	908
948	July 30·57	27	73	368	808	55	284	856
949	Aug. 26·80	27	137	754	656	90	523	769
950	Sept. 23·06	27	94	565	925	73	454	1053
951	Oct. 20·35	28	45	257	689	35	218	803
952	Nov. 16·65	27	129	836	683	98	663	824
953	Dec. 13·97	27	40	216	810	28	157	946

TABLE II.

Year.	No. of Days. Photographed.	Spotless.	Mean of Daily Areas. Projected.* Umbræ.	Whole Spots.	Faculæ.	Corrected for Foreshortening.† Umbræ.	Whole Spots.	Faculæ.
1913	365	311	2	10	87	1	7	95
1914	365	143	40	195	396	31	152	454
1915	365	6	186	920	1289	137	697	1521
1916	365	3	186	978	1512	135	724	1785
1917	365	0	346	2101	1882	247	1537	2305
1918	364	0	255	1504	1526	188	1118	1882
1919	365	0	253	1437	1445	182	1052	1729
1920	366	4	156	837	1052	112	618	1219
1921	365	31	110	571	647	79	420	739
1922	365	109	63	346	358	45	252	415
1923	365	171	15	74	189	10	55	222
1924	366	97	62	361	498	46	276	575

* Expressed in millionths of the sun's disc.
† Expressed in millionths of the sun's hemisphere.

TABLE III.

No. of Rotation.	Rotation commenced G.M.T. (Civil).	Spots NORTH of the Equator. Mean of Daily Areas.	Spots NORTH of the Equator. Mean Heliographic Latitude.	Spots SOUTH of the Equator. Mean of Daily Areas.	Spots SOUTH of the Equator. Mean Heliographic Latitude.	Mean Heliographic Latitude of entire Spotted Area.	Mean Distance from Equator of all Spots.
	1923. d.		°		°	°	°
940	Dec. 25·41	6	28·03	0	..	+28·03	28·03
	1924.						
941	Jan. 21·75	2	26·60	2	20·09	+ 1·85	23·15
942	Feb. 18·09	107	27·30	½	15·50	+27·14	27·26
943	Mar. 16·42	1	20·19	1	26·03	− 0·43	22·80
944	Apr. 12·71	36	25·78	142	28·17	−17·28	27·69
945	May 9·95	285	31·48	161	21·70	+12·29	27·95
946	June 6·16	309	30·60	14	24·92	+28·28	30·36
947	July 3·36	362	19·20	5	24·86	+18·56	19·28
948	July 30·57	275	15·32	9	23·80	+14·11	15·58
949	Aug. 26·80	494	18·67	29	25·24	+16·28	19·03
950	Sept. 23·06	295	20·84	159	27·83	+ 3·75	23·29
951	Oct. 20·35	216	21·66	2	25·86	+21·27	21·69
952	Nov. 16·65	663	21·35	1	23·05	+21·33	21·36
953	Dec. 13·97	74	20·77	83	24·24	− 3·08	22·61

TABLE IV.

Year.	No. of Days on which Sun photographed.	Spots NORTH of the Equator. Mean of Daily Areas.	Spots NORTH of the Equator. Mean Heliographic Latitude.	Spots SOUTH of the Equator. Mean of Daily Areas.	Spots SOUTH of the Equator. Mean Heliographic Latitude.	Mean Heliographic Latitude of entire Spotted Area.	Mean Distance from Equator of all Spots.
			°		°	°	°
1913	365	5	24·81	2½	20·08	+ 9·81	23·23
1914	365	99	21·11	53	23·06	+ 5·71	21·79
1915	365	379	17·95	318	19·75	+ 0·77	18·77
1916	365	470	15·14	254	17·05	+ 3·83	15·81
1917	365	860	13·82	677	15·66	+ 0·84	14·63
1918	364	609	11·97	509	13·69	+ 0·29	12·75
1919	365	559	9·85	493	11·79	− 0·28	10·76
1920	366	208	11·31	410	9·98	− 2·83	10·43
1921	365	261	7·27	159	8·92	+ 1·13	7·90
1922	365	161	8·82	91	6·62	+ 3·23	8·02
1923	365	33	13·88	22	17·32	+ 1·34	15·26
1924	366	232	22·19	45	25·54	+14·49	22·73

Table III. gives for each rotation in the year 1924 the mean daily area of the whole spots (corrected for foreshortening) and the mean heliographic latitude of the spotted areas for both hemispheres, northern and southern. The mean heliographic latitude of the entire spotted area and the mean distance from the equator of all spots are also tabulated. The mean values for the years 1913 to 1924 are given in Table IV. Tables II. and IV. are in continuation of similar tables for the years 1874 to 1888 on pp. 381 and 382 of vol. xlix of the *Monthly Notices*; for the years 1889 to 1902 on pp. 465 and 466 of vol. lxiii.; and for the years 1901 to 1914 on pp. 402 and 403 of vol. lxxvi.

The rotations in Tables I. and III. are numbered in continuation of Carrington's series (*Observations of Solar Spots made at Redhill*, by R. C. Carrington, F.R.S.), No. 1 being the rotation commencing 1853 November 9. The assumed prime meridian is that which passed through the ascending node at mean noon of 1854 January 1, and the assumed period of the Sun's sidereal rotation is 25·38 days.

The principal features of the record for the year 1924 were as follows:—

1. A decided increase in the mean daily area of spots—276 units as compared with 55 units for the previous year, which is thus the minimum year of the passing cycle.

2. Of the total spot activity, only 16 per cent. was contributed by the southern hemisphere.

3. The amount of faculæ showed a well-marked increase, especially in the latter half of the year.

4. The mean latitude of all spots was 23°. Spots in equatorial regions were infrequent and small, with the notable exception of Group 9552 (lat. 5° N.) and its subsequent returns as Groups 9558, 9565, and 9570. The highest latitude recorded during the year was that of a small spot on July 23 at 46° N.

5. There were 97 spotless days compared with 171 in the previous year. Seventy per cent. of the spotless days occurred in the first three months of the year.

6. The number of separate groups lasting for at least two days was 75 as compared with 33 for the year 1923. Of these 75 groups, 54 were in the northern hemisphere and 21 in the southern. Besides these, there were recorded 48 small groups (mostly consisting of one or two very small spots) visible on one day only; 35 were in the northern and 13 in the southern hemisphere.

7. The year's record accordingly showed the minimum to be definitely past and the growing activity of the new cycle well established.

Royal Observatory, Greenwich
1925 *August* 31

Note on the Personality of Mr. C. Grover in Estimating Differences of Stellar Magnitude at the Rousdon Observatory in the Years 1885–1920. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The Rousdon observations of variable stars 1885–1900 have been published in *Mem. R.A.S.*, vol. lv. On pp. x–xvii of the introduction to that volume attention is drawn to a peculiarity in Mr. Grover's estimations of magnitude differences. They do not bear a constant ratio to the actual differences except (presumably) for a small range of difference between +0·4 and −0·4. Outside these limits they numerically exceed the actual differences by a sensibly constant amount—constant, that is, for stars near the same mean brightness, though the constant itself is smaller when the stars are faint. Most variable star observers restrict themselves in practice to the estimation of differences within the limits +0·4 to −0·4, so that the study of Mr. Grover's observations, wherein differences up to two or three magnitudes are estimated, has little direct bearing on the great mass of observations. The Rousdon procedure was quite exceptional and is open to various objections. Still, the mere fact that it is not likely to be adopted elsewhere, added to the fact that it was adopted in practice at Rousdon by the same observer over a period of a third of a century, makes it worthy of attention as an extreme case; for the peculiarities of extreme cases sometimes throw light on those of intermediate cases which are not so fully developed.

2. For instance, it is a matter of importance to all variable star observers to know whether their own personal habits of observation are likely to change in course of time. When they restrict their estimations within small limits, of a few tenths of a magnitude, it may not be very easy to detect a change in the ratio of a "step" to the actual magnitude difference 0·10; for probably only a few hundredths of a magnitude are concerned in any case. But Mr. Grover's personality runs up to 0·2 or 0·3 at least, over a large part of the observational range; and changes in differences of this kind should be easier of detection. Indeed, one such change has already been noted (p. xv, *loc. cit.*); the years 1898–1900 differ sensibly from 1893–1897, though before 1893 the habits of 1898–1900 reappear. It is thus possible to take two views of the personality: (*a*) that it is essentially variable—liable to changes progressive with time, though not always in the same direction; (*b*) that it is essentially constant, but liable to temporary disturbance (as in 1893–1897) by some causes not yet understood.

In either case the examination of the years after 1900 is clearly of interest.

These observations were put into my hands after Mr. Grover's death, and discussion of them has proceeded as opportunity offered; and though it has suffered from regrettable delays (see § 4), it is hoped to complete the results for some individual stars very shortly. The examination of the personality is an obvious preliminary.

3. The procedure is described in the above introduction. The magnitude of the variable is adopted as the mean from five comparisons:

each individual comparison then, by inversion, gives a magnitude for the comparison star ; and these deduced magnitudes are collected under the heading of the interval (whether ±0·1 or ±2·0, etc., magnitudes) assigned by Mr. Grover between variable and comparison star in that particular case. The essential facts are collected in Table I., which corresponds to Table III. on p. xii of the Memoir cited ; but some of the extreme groups have been omitted, and the mean corrections to magnitude have been given to 0·01 instead of 0·1 only as before.

4. The intervals in the first column are those recorded by Mr. Grover between the variable and the comparison star, the minus sign denoting that the variable is the fainter. Thus the record $a-1{\cdot}0$ was used to mean that the variable is a magnitude fainter than a. The opposite convention was used by Knott and Pogson (*Mem. R.A.S.*, **58**, p. xi) and has the obvious advantage that the magnitude of the variable is indicated algebraically. Pogson further used the convenient unit of 0·1 magnitude, so that if star a were of magnitude 7·5, he wrote 75 instead of a : and the record $a-1{\cdot}0$ would become $75+10=85$. This plan is so simple that I thought at one time of printing the Rousdon observations subsequent to 1900·0 in this way, in spite of the fact that those before 1900·0 are already printed with the other signs ; and indeed the copy for press was actually prepared in Pogson's notation. But I found that variable star observers prefer the Rousdon notation, and it seems undesirable to make the change now, though I cannot but think that it is unfortunate that it was not made earlier. The hesitation I felt as to the better plan to adopt has been largely responsible for the delay in publication ; and even now I am tempted to evade the decision by printing simply the deduced magnitudes, leaving the details in MS. in the R.A.S. library.

5. The first entry in the table under heading]8·5, viz.

$$-2{\cdot}0 \quad 57 \quad -38$$

means that 57 records were used where the variable was recorded as 2 magnitudes fainter than some comparison star brighter than 8·5 mag., and that in the mean of these 57 Mr. Grover's estimate of 2·0 should have been 1·62, being always numerically too large.

6. Two sets of means are taken, in columns and in rows. Consider first those of the columns. They show that the error of record is largest for bright stars and smallest for faint. Incidentally this explains why the second set of means is larger than the first : for at the top of the table the comparison is made between a star of magnitude (say) 8·0 and the variable 10·0 (2 magnitudes fainter)—mean of the two 9·0 ; whereas at the foot of the table the comparison is between the same star 8·0 and the variable 6·0 (2 magnitudes brighter)—mean of the two 7·0. There is thus an interval of 2 magnitudes between these two extreme cases. For the means of the two halves of the table this difference is reduced to about one magnitude. This difference is responsible for the excess of the positive means (foot of table) over the negative (half-way line), by about 0·05 at the bright end and 0·01 at the faint end : 0·02 in the weighted mean. More precise discussion is scarcely justified.

TABLE I.

Collected Results for Corrections to Adopted Magnitudes of Comparison Stars deduced from the Observations.

(Unit in body of table 0·01 magnitude.)

	]8·5.		8·5 to 8·9.		9·0 to 9·5.		9·6 to 10·1.		[10·1.		Total.	Mean
Recorded Interval.	No. Obs.	Corr.	No. Obs.	Corr.	No. Obs.	Corr.	No. Obs.	Corr.	No. Obs.	Corr.	No.	(Wt.
−2·0	57	−38	36	−34	28	−31	63	−24	12	−36	196	−3
−1·7	9	−36	15	−17	11	−20	18	−24	5	−14	58	−2
−1·5	37	−21	48	−25	35	−20	46	−16	16	− 8	182	−1
−1·4	8	−20	11	−22	12	−18	6	−28	4	+ 5	41	−1
−1·3	64	−15	59	−28	67	−10	65	−14	27	−11	282	−1
−1·2	39	−26	76	−24	56	−23	61	−23	30	−15	262	−2
−1·0	74	−44	121	−25	189	−11	176	−16	169	−12	729	−1
−0·9	2	−10	6	−17	4	0	9	− 1	7	−26	28	−1
−0·8	7	−26	44	−16	63	−15	53	− 7	55	−13	222	−1
−0·7	30	−38	41	−24	58	−11	55	−11	56	− 9	240	−1
−0·6	26	−26	42	−14	67	−12	59	−13	51	−13	245	−1
−0·5	28	−15	40	− 8	64	− 9	84	− 5	102	− 2	318	−
−0·4	7	−13	19	−14	45	− 3	37	− 7	37	− 2	145	−
−0·3	36	− 7	60	−19	117	− 9	126	− 4	119	0	458	−
−0·2	23	− 9	46	− 7	125	−11	131	− 9	118	− 4	443	−
−0·1	10	+ 8	25	− 4	52	−12	90	−10	92	− 5	269	−
Mean	457	−27	689	−20	993	−13	1079	−11	900	− 8	4118	−1
+0·1	9	+12	17	+12	40	+ 6	75	+ 2	77	− 6	218	0
+0·2	15	+21	50	+15	76	− 2	89	+ 6	92	− 4	322	+0
+0·3	32	+20	48	+19	101	+ 8	121	+11	85	+ 5	387	+1
+0·4	10	+55	18	+27	38	+ 6	47	+ 9	39	− 3	152	+1
+0·5	18	+22	36	+27	87	+12	71	+18	80	+ 7	292	+1
+0·6	20	+25	34	+29	61	+14	79	+ 9	60	+ 6	254	+1
+0·7	27	+31	40	+32	98	+21	82	+20	62	+10	309	+2
+0·8	30	+20	50	+30	71	+18	69	+20	78	+11	298	+2
+0·9	1	+20	5	+54	10	+ 8	15	+13	17	+ 3	48	+1
+1·0	94	+34	126	+29	272	+23	269	+17	213	+13	974	+2
+1·2	14	+46	45	+41	80	+26	110	+14	62	+11	311	+2
+1·3	40	+29	47	+38	129	+22	135	+14	73	− 1	424	+1
+1·4	4	+48	8	+25	29	+39	25	+ 5	19	0	85	+1
+1·5	38	+34	32	+30	130	+29	112	+16	62	+ 8	374	+2
+1·7	20	+45	19	+33	49	+29	78	+34	49	+ 9	215	+2
+2·0	33	+30	27	+ 1	75	+18	169	+12	192	+20	496	+1
Mean	405	+31	602	+26	1346	+18	1546	+12	1260	+ 7	5159	+1

7. Now take the means of the rows as exhibited in the last column. It will be seen that, except for the intervals near the middle of the table (say −0·6 to +0·6), the means are nearly constant. Taking positive and negative together the simple mean of those from 1·3 to 2·0 is 21; and of those from 0·7 to 1·2 is 18, very slightly smaller. This constancy of the error for intervals exceeding half a magnitude was the most striking feature of Mr. Grover's personality, and it seems to have remained sensibly unaltered in his later years.

8. There is, however, one detail worthy of notice. In the upper half of the table the differences tend to be larger for the larger intervals, in the lower half smaller; and we have obtained the above constancy only by setting the increase against the decrease. But this is not unfair, for these departures from constancy are largely spurious, being due to the preponderance in the former case of bright, and in the latter of faint, stars. Take, for instance, the extreme rows for interval ±2·0. The numbers of available comparisons are:

Interval.	]8·5.	8·5 to 8·9.	9·0 to 9·5.	9·6 to 10·1.	[10·1.
−2·0	57	36	28	63	12
+2·0	33	27	75	169	192

In the upper row the first two groups contain nearly as many stars (93) as the other three: in the lower they are only 60 to 436.

9. A more elaborate discussion might perhaps be made, for it is astonishing how accurately these estimates of intervals of one or two magnitudes (or even more), which must be very rough in the individual cases, behave in the mean of a number. But the above study is amply sufficient for the main purpose, viz. to justify the continuance of the rules of reduction adopted in vol. lv. for the observations made before 1900 to those made subsequently.

Summary.

An examination of a considerable portion (not the whole) of the Rousdon observations made from 1900 to the death of Mr. Grover shows that his personality in estimates of intervals up to 2 or 3 magnitudes remained sensibly the same as found in *Mem. R.A.S.*, **55**, pp. x–xvii, viz. his mean intervals are in excess of the true interval in magnitudes by about 0·20 magnitude for all intervals exceeding half a magnitude. But this mean value 0·20 must be increased when the stars compared are bright, and diminished when they are faint: though in both cases it remains steady over a considerable range of interval when the mean value of the pair compared also remains steady. The rule of reduction (*c*) given at the foot of p. xvi, *loc. cit.*, is therefore applicable to the later observations.

The Forms and Development of the Spiral and Allied Nebulæ.

By J. H. Reynolds. (Plates 20, 21.)

The form of the spiral nebulæ was investigated first in 1911 by Von Pahlen (*Ast. Nach.*, Bd. 188). He dealt with the three spirals M33, M74, and M51, and after allowing for the projection of the spiral plane on the celestial sphere, he found the spiral curves to be of the form $r=xe^{\text{const.}\,\phi}$, x being a constant expressed in the linear units used in measuring the photographs, and ϕ the position angle. This is the formula for equiangular or logarithmic spirals, and the calculated and observed points at various position angles were found to give only small residuals. The tangential angle of the spiral arms with the radius vector was, however, not found to be the same even in the arms of the same spiral. Thus for the eastern branch of M33, $r=1{\cdot}2e^{\cdot01490\phi}$, and for the western branch $r=1{\cdot}2e^{\cdot01398(\phi-165^\circ)}$. In the former 220° of a complete revolution was taken, in the latter 240°. This spiral is, however, not suited to such exact measurements as were made, as, although the general form is roughly spiral, it is of the massive type and the arms are broad and irregular. In M74, which is filamentous in structure and therefore much better suited to the investigation, one arm is a well-developed spiral curve for almost two revolutions, and 530° were compared with the formula $r=1{\cdot}7e^{\cdot00495\phi}$; the other arm soon breaks into irregularities of structure and only 330° were compared with the formula $r=1{\cdot}7e^{\cdot06428(\phi-145^\circ)}$. The tangential angle in M74 is thus considerably greater than that found for M33. The other spiral dealt with in the same manner was M51, which gave the formula $r=25e^{-\cdot00408\phi}$. This spiral is of filamentous type and also develops considerable abnormalities in the curve.

A paper by Dr. Groot has recently been published (*M.N.*, **85**, 535) in which the author has extended Von Pahlen's method to other spirals. The conclusion reached is that in eight cases out of the nine dealt with, the curve of the spiral is approximately equiangular. The method of measurement adopted and the objects chosen for measurement do not seem capable of giving anything more than a rough determination, as he measures the same object (reproduced in two different publications, as N.G.C. 4321 and M100 respectively) twice over with different results, without recognising that he was dealing with one object only. The choice of spirals considerably inclined to the normal to the line of sight also introduces uncertainties. It is impossible to fix the inclination of the spiral planes nearer than 10° to the normal owing to the varying tangential angles of the arms and bifurcations, which do not permit of a rigid mathematical framework being adopted depending on the sine of the inclination.

If a number of photographs of spiral nebulæ are taken, it will be found that the commencement of the curve is in most cases approximately of the logarithmic spiral form, but the tangential angle of the curve with the radius vector varies considerably, some spirals being much more closely wound than others. The angular development also varies very much, some spirals not being developed for more than half a

revolution, others completing almost two revolutions. In the latter case great irregularities have invariably developed, and the original curve, although still existing near the nucleus, becomes much distorted, the angle in some cases decreasing or in other cases increasing rapidly with a resulting asymmetry with regard to the nucleus. The amount of development of the spiral arms must be taken in connection with the formation of condensations as an indication of the age of the spiral. As the progress of development of the spiral arms is much more definite and apparent in the filamentous spirals, the examples which have been selected for detailed examination are of this type; and as those where the plane is nearly tangential to the celestial sphere are more readily dealt with than those of considerable inclinations, the number which can be considered is not great. Six have, however, been selected so as to form a series from early to advanced development.

The massive spirals usually do not show the spiral form as well as the filamentous, and it is sometimes difficult to place them in any series owing to abnormal developments. One feature of the massive spirals—the large and usually bright amorphous nucleus—is never met with in the filamentous type. The original nuclei from which the massive spirals such as M31, M33, M81 and many others were originally derived were probably much larger and brighter than any of the existing spheroidal nebulæ. It seems as if the development of the spirals from the original nuclei has already taken place in the great majority of cases, as examples of rudimentary spiral arms are very rarely found. The spheroidal nebulæ are, no doubt, of sufficient mass to develop spiral arms, but the many thousands of faint nebulous spots resembling diffuse star discs seem to be of another class altogether. Their appearance in groups suggests that they come at the end rather than at the beginning of the series, and are possibly the remains of condensed spiral nebulæ which have exhausted their nuclei and become spread over a large area, some condensations appearing as faint star discs, others as still nebulous spots. Besides the usual spiral form there are other forms where the spiral arms do not originate in the nucleus itself, but at the ends of a straight luminous band symmetrically disposed on each side of the nucleus. Such spirals have been termed ϕ spirals in *L.O. Pub.*, **13**, and Hubble has proposed the more suitable term "barred spirals." Some of these have the appearance of the letter "S," others of an elliptical composite ring with a luminous band along the major axis. Attention was drawn to this type by Knox Shaw in *H.O.B.*, **15**, and many examples are known, such as N.G.C. 1433, 1512, 4725, and 5850. It is difficult to explain why the luminous band should almost invariably extend along the major axis of the ellipse, unless it is an optical effect caused by the heaping up of luminous material in the line of sight. In this case it would not be a straight band of matter extending each side of the nucleus, but a lenticular-shaped mass of approximately circular form, co-extensive with the ring. It is doubtful, however, if the same explanation could be given of the band observed in the "S"-shaped spirals. Outside the composite ring, spiral filaments appear in some cases, so this form may be classified as a special type of spiral where the lenticular nucleus has been developed to an

abnormal degree. The comparatively small angular size of these formations (5′ is probably a maximum) is in accordance with the suggested explanation.

A detailed examination of the development of the spiral and allied forms of nebulæ must of necessity have reference to Jeans' theoretical work on the subject in his *Problems of Cosmogony*. The general confirmation of his hypothesis in the various examples is obvious to anyone who has studied the subject.

If we start from Jeans' theoretical lenticular form, of which a limited number are known as edge-on objects (N.G.C. 5868), the next stage to be expected is that in which two nodules of matter form at opposite ends of a diameter of the lenticular-shaped nucleus. There seems to be no objects yet photographed which give this characteristic form, unless N.G.C. 2859 can be so interpreted. It is possible that some may exist undetected amongst the small nebulæ, but we have to pass on to the next stage, where the arms have already developed their spiral character. Examples where the spiral is only developed about half a revolution from the nodal point are rare, but they exist. N.G.C. 5427 (*H.O.B.*, **22**, pl. v) shows a bright, fairly large nucleus with two spiral arms extending in one case a little more, in the other case a little less, than half a revolution. As the total angular diameter of this open spiral is only 2′, it is not suitable for diagrammatic treatment. A similar object is reproduced in *Mount Wilson Contributions*, **7**, pl. ix (N.G.C. 2681). It is described as a single arm, but it seems possible that two arms may be present, one of which appears to be cut off near the nucleus by absorbing matter. The presence of a faint outer ring, however, makes the object rather anomalous. Both these spirals are also abnormal in the large angle made by the tangent to the curve with the radius vector (about 80°).

The following method of producing the diagrams (Plate 20) given in this paper was adopted. Glass positives of the photographs were projected optically on diagrams on which circles of equally increasing radii had been described with position angles. In cases where the plane of the nebula is not tangential to the celestial sphere, the angle of projection has been suitably inclined to make the planes approximately coincident. The spirals of small angular diameter have been enlarged so that the scale of the spiral is roughly comparable.

The best example available of a little developed spiral is N.G.C. 5247, which is now dealt with in detail with others. In the following descriptions r=radius in arbitrary units, ϕ=position angle, θ the angle made by the tangent to the curve with the radius vector, ω the inclination of the plane, and Δ the dimensions.

N.G.C. 5247 (*Helwan*).

$\lambda=284°$, $\beta=+45°$, $\omega=\pm 90°$, $\Delta=5'\times 5'$.

A typical filamentous spiral with small bright nucleus and isolated condensations in the outer regions. The southern arm A is developed over 250°, with a bifurcation A1, ending in a condensation at 280°.

Fig. 1. N.G.C. 5247.

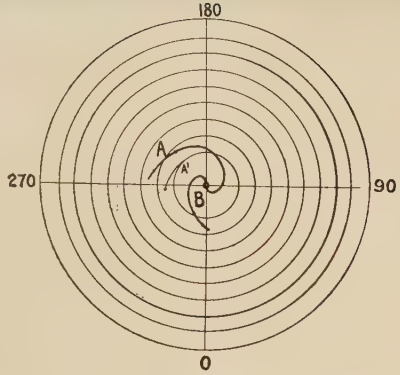


Fig. 2. N.G.C. 2835.

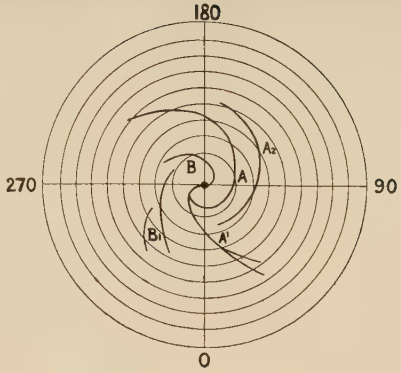
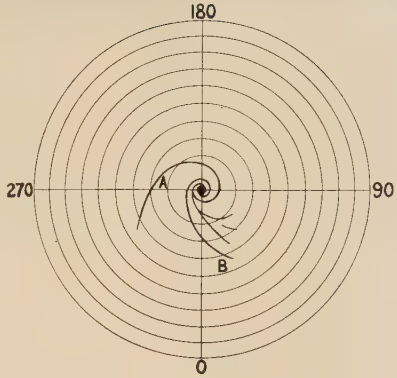


Fig. 3. N.G.C. 4254.



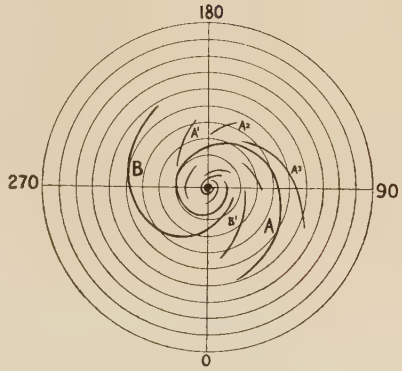


Fig. 4. N.G.C. 4321.

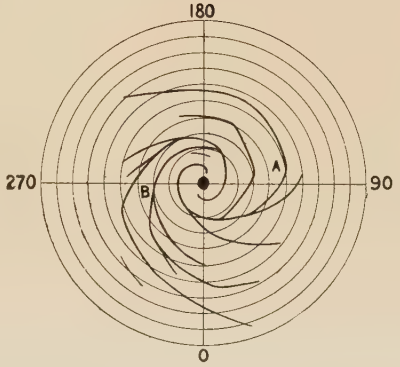


Fig. 5. N.G.C. 1232.

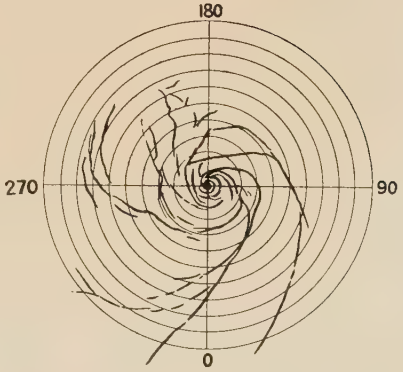


Fig. 6. N.G.C. 5457.

The northern arm B only extends for 180°. The plane of the nebula is approximately tangential to the celestial sphere.

	r.	ϕ.	θ.		r.	ϕ.	θ.
		°	°			°	°
A.	1	90	58	B.	1	270	68
	2	180	63		2	338	58
	3	240	65				
A1.	2	253	63				

N.G.C. 2835 (*Helwan*).

$\lambda=220°$, $\beta=+21°$, $\omega=\pm75°$, $\Delta=6'\times5'$.

This spiral consists of two principal arms arising normally from diametrically opposed nodes in the nucleus, but both develop important bifurcations which give the appearance of a 5-armed spiral. The outermost bifurcation A2, towards $\phi=90°$, is apparently disconnected with the nucleus, but originated presumably in a branch of A1. In no case does a spiral arm complete a revolution, but the A system is much more developed than the B. The A system also contains several condensations in the outer regions.

	r.	ϕ.	θ.		r.	ϕ.	θ.
		°	°			°	°
A.	1	292	43	B.	1	155	52
	2	103	58		2	204	58
	3	152	60				
	4	178	60				
	5	203	67				
A1.	2	341	45	B1.	3	298	65
	3	5	46		4	323	54
	4	22	40		5	336	50
	5	25	50				
	6	38	60				
A2.	3	90	71	B2.	4	298	40
	4	126	71		5	312	62

N.G.C. 4254 (*Lick*).

$\lambda=240°$, $\beta=+76°$, $\omega=\pm65°$, $\Delta=4{\cdot}5'$.

A little developed but much condensed spiral, with an important bifurcation of the B branch near the nucleus which gives the effect of a 3-branched spiral. B1 after a normal spiral form bends back towards the nucleus, and there are suggestions of the same in B. A is a simple curve without branches. The presence of numerous well-defined condensations quite close to the nucleus makes the position of the

nodes of origin uncertain, and it is possible that the bifurcation of the B branch should be attributed to the A branch. Curves of condensations occur between, which are difficult to attribute definitely to either branch.

	r.	ϕ.	θ.		*r.*	ϕ.	θ.
		°	°			°	°
A.	2	222	62	B.	1	304	60
	3	268	60		2	343	45
	4	295	52		3	3	45
					4	21	50
				B1.	1	352	50
					2	40	55

N.G.C. 4321 (*M*100).

$\lambda=238°$, $\beta=+78°$, $\omega=\pm 80°$, $\Delta=5'\times 5'$.

A much bifurcated spiral developed through at least one and a half revolutions with many condensations in the outer regions. The nuclear regions are elongated in ϕ 130°–310° and indicate an inclination of the spiral plane of 70° to 60° to the line of sight. It is difficult to reconcile the outer regions with this inclination, and the measures are therefore given without any corrections for inclination. The values of θ for A are in general agreement with the value of β derived by Dr. Groot.

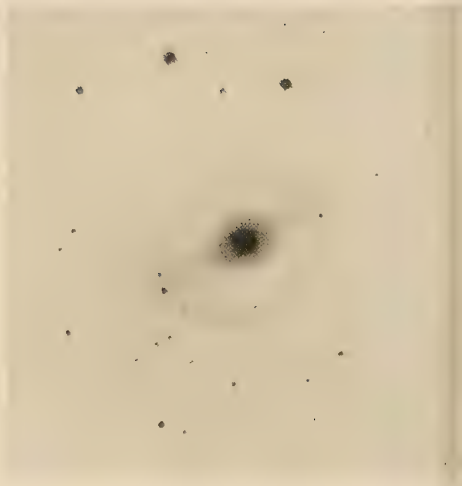
	r.	ϕ.	θ.		*r.*	ϕ.	θ.
		°	°			°	°
A.	2	310	85	B.	2	28	67
	3	162	80		3	342	56
	4	113	72		4	310	60
	5	60	72		5	258	68
A1.	3	217	50	B1.	2	93	55
					3	45	61
					4	15	50
A2.	4	163	63				
A3.	4	138	60				
	5	107	62				
	6	86	53				

N.G.C. 1232 (*Helwan*).

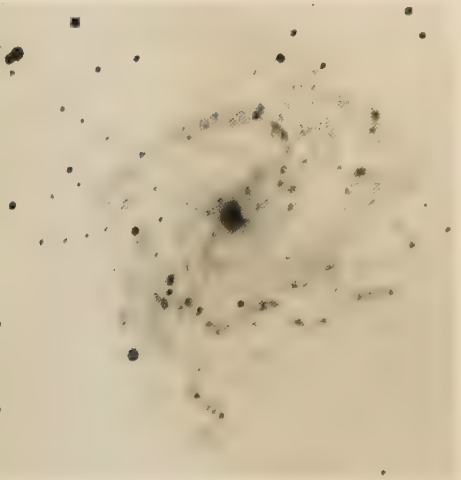
$\lambda=178°$, $\beta=-58°$, $\omega=\pm 80°$, $\Delta=6'$.

The development in this case is considerably more than a revolution for each arm. Both arms contain many important bifurcations with large irregularities of form and angle, so irregular in fact that a detailed

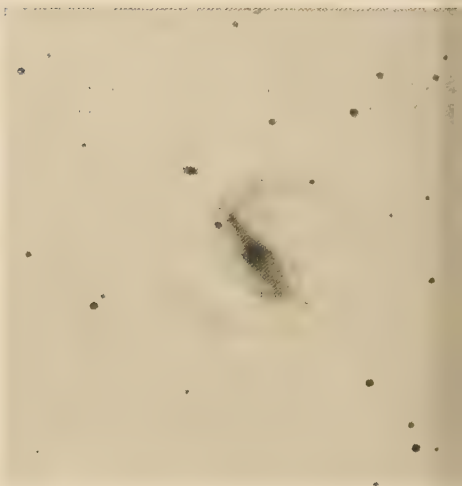
N.G.C. 4631



N.G.C. 5850



N.G.C. 5236



N.G.C. 1097

measurement would be of no advantage. The general form is shown in fig. 5, and an inspection of this gives a better idea than measures would do. Many condensations appear in the spiral in accordance with its advanced stage of development. The arm A, with its branch A1, is of constantly decreasing angle ($\theta=58°$ to $41°$). A2 branches off the main arm A at a considerable angle and is almost straight. A short band is succeeded by another straight length, followed by a curve where the value of ϕ is about 80°. A similar form is described by the continuation of A to a large irregular condensation at $\phi=230°$. Similar irregularities occur in B, which, however, is not so far developed as A.

N.G.C. 5457 (*Mount Wilson*).

$\lambda=68°$, $\beta=+58°$, $\omega=\pm 90°$, $\Delta=18'$.

This is the well-known spiral M101, and is chosen as an example of what seems to be the break-up of the spiral form. Its irregularities of structure are so great that, even more than in the last example, detailed measures are of no value. A conspicuous feature is the asymmetry with regard to the nucleus. In $\phi=135°$ it only extends 4′ from the nuclear point, while on the opposite side it reaches 14′. Although it is difficult to give a definite estimate of the angular development of the spiral owing to its great irregularity, it seems to have accomplished almost two revolutions. Its asymmetrical form suggests that the plane of the spiral is also breaking up, and the possibility of this is confirmed by the condensed edge-on spirals such as N.G.C. 4631, which no longer show the evenly disposed form usually exhibited by these objects in the earlier stages of development.

In plate 21 three photographs of the " barred " and " S "-shaped spirals are reproduced in negative in addition to N.G.C. 4631 (fig. 1). It will be noted that the band of absorption along the axis, usually so conspicuous in edge-on spirals, has in this example disappeared almost entirely, the dark matter having presumably been taken up by the condensations. The irregularity of the plane is obvious.

Fig. 2 is a reproduction of the little-known object N.G.C. 5850 from Mount Wilson, and shows very well the " bar " of illumination which apparently marks the major axis of the ellipse. Outside the ellipse are faint spiral filaments. The nuclei of these and of the " S "-shaped spirals are usually bright and well defined, and divided up into three or more sections if short exposures are given.

Fig. 3 (N.G.C. 5236) has been photographed in America and is reproduced in *L.O. Pub.*, **13**. The present photograph from the Helwan Observatory shows remarkable detail and extension, including absorption markings near the nucleus, and a great number of stellar condensations which obviously belong to the nebula and not to the star field. It is certainly one of the most interesting spirals in the sky. The question of the relative size and distance of this spiral compared with the great spirals M31 and M33 is important, as the condensations appear to be stars and not star clusters. As Dr. Hubble has included this nebula in his programme of work on the Cepheid variables, we may soon get some information of a definite nature on this point.

Fig. 4, N.G.C. 1097, reproduced from a Helwan photograph, is also a little-known object, although large in angular size and of unusual structure. It consists of a composite nucleus with an irregularly curved and streaked bar, from the extremities of which issue two spiral arms of different curvatures and an elongated small nebula which may or may not be connected with the main body.

Conclusion.

The foregoing examples, which can be taken as typical, are sufficient to show the general state of development of the spirals. The great majority of both filamentous and massive spirals are evidently in an advanced state of development, as far as we can judge from their present aspect, which of course is only a relative one depending on their distance, now known to be very great. Spirals developed for less than half a revolution are very rare, and, on the other hand, spirals developed for more than two revolutions in an actually spiral form are unknown. We know many examples of the nuclear spheroidal form from which the spirals in all probability have developed, but we can only conjecture what has happened in cases where the spiral form and plane has ultimately broken up and the nucleus become exhausted. Whether the so-called "nests" of small nebulæ are the remains of broken-up spirals it is impossible to say, but it is remarkable that these are often found in close proximity to the Galaxy (see *M.N.*, **84**, 76).

The angle usually made by the tangent of the curve with the radius vector is usually between 50° to 70° in the earlier part of the curve, but in more advanced stages the angle may be anything between 40° and 80°. In many cases these extremes may be observed in the two arms and bifurcations of the same spiral. The bifurcations of one of the arms are usually more numerous than the other, a feature to be noted in little developed as well as much developed objects. So asymmetrical does the usual development of the two arms appear with regard to the nucleus, that it is a matter of surprise that the form of spirals seen edgewise is in most cases remarkably symmetrical in this respect. It may be that the filamentous spirals are more readily broken up by internal stress than the massive type, although in the latter great irregularities of structure are often apparent, *e.g.* N.G.C. 4258.

Micrometrical Measures of Double Stars, made with the 10-in. Refractor of the Armagh Observatory.

(*Communicated by the Director.*)

The following measures of double stars were made by Mr. M. A. Ellison :—

Star.	1920. R.A.	Dec.	P.	D.	Mags.	No. of Nights.	Date. 1925. +
	h m	° ′	°	″			
Σ 93 (Polaris)	1 31·7	+88 53	216·7	18·18	2, 9	2	·186
Σ 612	4 49·9	+ 7 15	198·0	16·28	7·6, 7·9	2	·160
Σ 627	4 56·4	+ 3 30	260·4	21·19	6·8, 7	2	·160
Σ 738 (λ Orionis)	5 30·7	+ 9 53	43·1	4·31	5, 6	3	·138
η Orionis	5 20·5	− 2 28	83·8	1·28	4, 5	2	·213
Bowyer	5 29·03	+26 54·6	292·0	4·15	10·2, 10·5	2	·181
Σ 749	5 31·9	+26 53	342·4	1·00	7·1, 7·2	2	·187
Σ 763	5 34·9	+10 14	317·8	6·08	8·2, 8·8	3	·170
Σ 774 (ζ Orionis)	5 36·7	− 1 58	159·0	2·54	2, 5·7	2	·187
Σ 816	5 50·6	+ 5 51	288·8	4·39	6·2, 8·7	2	·209
Σ 820	5 51·3	+ 8 59	111·1	4·96	8·3, 8·8	2	·196
Σ 1110 (α Geminor.)	7 29·5	+32 4	212·7	4·77	2·7, 3·7	4	·167
OΣ 175	..	..	328·8	0·84	6, 6·6	2	·203
OΣ 179 (κ Geminor.)	7 39·6	+18 15	236·2	6·88	4, 8·5	2	·206
Σ 1196 (ζ Cancri)	8 7·6	+17 54	242·2	0·66	5·0, 5·7	2	·224 A.B.
			106·9	5·46	5·0, 5·5	2	·224 A.C.
Σ 1332	9 12·7	+23 59	23·4	5·82	7·2, 7·5	2	·178
Σ 1536 (ι Leonis)	9 19·7	+10 59	32·6	1·49	3·9, 7·1	4	·250
Σ 1353	9 23·3	+16 6	309·5	3·16	8·5, 8·8	3	·199
Σ 1355	9 23·6	+ 6 36	337·1	2·59	7·2, 7·2	2	·220
Σ 1356 (ω Leonis)	9 24·2	+ 9 25	132·6	1·04	6·2, 7	2	·199
Σ 1389	9 47·8	+27 22	304·8	2·36	8, 9	2	·220
Σ 1413	10 8	+16 44	275·6	2·12	8·9, 8·9	2	·311
Σ 1417	10 10·8	+19 31	259·8	2·29	8·2, 8·2	2	·308
Σ 1424 (γ Leonis)	10 15·5	+20 15	118·3	3·90	2, 3·5	5	·287
Σ 1450 (49 Leonis)	10 30·9	+ 9 4	157·2	2·36	6, 8·7	2	·308
Σ 1487 (54 Leonis)	10 51·3	+25 10	107·9	6·30	5, 7	3	·324
Σ 1504	10 59·9	+ 4 4	290·3	1·28	7·5, 7·6	2	·366
Σ 1527	11 14·8	+14 43	21·8	2·83	6·9, 8·1	2	·312
Σ 1537	11 20·3	+20 59	355·0	2·28	7·6, 8·6	2	·330
Σ 1785	13 45·5	+27 23	39·5	1·04	7·2, 7·3	3	·367
Σ 1793	13 55·4	+26 13	242·2	4·52	7, 8	2	·338
Σ 1794	13 56	+20 17	127·4	1·89	8·5, 8·7	3	·354

Occultations of Stars by the Moon, observed at Tomsk University Observatory during the years 1924, 1925. By N. Goryatchew.

(*Communicated by the Secretaries.*)

($\phi = 56° 28' 07''$; $\lambda = 5^h 39^m 47^s{\cdot}4$ east of Greenwich.)

Date.	Star.	Magn.	Phase.	Local Sidereal Time.
1924.				h m s
Sept. 16	ξ^2 Ceti	4·3	D.	02 37 27·1
	ξ^2 Ceti	4·3	R.	03 07 31·6
Oct. 16	θ^1 Tauri	4·2	D.	23 17 39·9
	θ^2 Tauri	3·6	D.	23 31 54·8
	θ^2 Tauri	3·6	R.	23 55 18·3
	θ^1 Tauri	4·2	R.	00 11 12·4
	264 B Tauri	4·8	D.	00 20 33·2
	B.D. +15° 633	6·6	R.	00 37 06·6
	264 B Tauri	4·8	R.	01 29 00·7
	275 B Tauri	6·5	R.	03 18 03·6
	α Tauri	1·1	D.	03 37 51·9
	α Tauri	1·1	R.	04 54 02·6
Nov. 4	136 B Capricorni	7·1	D.	00 36 07·2
1925.				
Feb. 3	75 Tauri	5·2	D.	10 46 53·1
15	8 G. Libræ	6·9	R.	13 23 29·6
Mar. 1	179 B Tauri	5·9	D.	10 01 56·3
16	49 Libræ	5·4	R.	13 03 01·0
31	68 Orionis	5·7	D.	07 40 43·6
Apr. 4	7 Leonis	6·2	D.	08 59 57·0
	11 Leonis	6·5	D.	10 13 51·6
5	ψ Leonis	5·6	D.	13 48 23·8
May 1	227 B Cancri	6·4	D.	12 46 38·9
13	253 B Sagittarii	6·1	R.	17 50 23·1

Observer.—1924 Sept. 16 (ξ^2 Ceti) and 1924 Nov. 4 (136 B Capricorni), Mrs Gort-de-Grott; other occultations were observed by Prof. N. Goryatchew.

Instrument.—Zeiss telescope, 90 mm. aperture; magnifying power from 90 to 150. Observations made with eye and ear.

Errata.

P. 707, Equation (67), last line, *for* $-{\cdot}3973800 = 0$ *read* $-{\cdot}3973800\ c_3 = 0$.
P. 710, line 11 from bottom, *for* ν_1 ·002186 *read* ν_3 ·002186.
P. 710, last line, *for* $l_1|$ *read* $|l_1|$.
Pp. 729–731, Tables II., III., IV., heading of column 1, 6, etc., *for* c_1°, c_3°, *read* c_1^0, c_3^0.

LIST OF ADDITIONS

TO THE LIBRARY

OF THE SOCIETY

1924 JUNE 30 TO 1925 JUNE 30.

Serial Publications received in the regular course, and Reprints of Papers published in some of these Serials, are not included in this Annual List.

An asterisk (*) indicates that the work is an excerpt.

Algiers, Observatoire:

Carte photographique du Ciel. Zones +1°, +3° (8 charts).
(*Observatory.*)

Amsterdam, Astronomical Institute of the University:

Publications.
(*Institute.*) 4to. Amsterdam, 1924

No. 1. A. Pannekoek, Researches on the Structure of the Universe. 1. The Local Star-system deduced from the Durchmusterung Catalogues.

Araki (Toshima):

*Ueber die Beziehung zwischen der Periode und dem Umfange des Lichtwechsels der Cepheiden und der langperiodischen Veränderlichen Sterne.
(*Author.*) Large 8vo. [Kyoto], 1924

Barcelona, Sociedad Astronomica de España y America:

Revista. Año 14, Num. 98.
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(*Lowell Observatory.*) 8vo. Northfield, 1924

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Die Geschichte der Zeitmessung und der Uhren.
(*Purchased.*) 4to. Berlin u. Leipzig, 1925
Band I., Lieferung E. Joseph Drecker, Theorie der Sonnenuhren.

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*Über die Laplacesche Methode der Bahnbestimmung im Vergleich zur Gauss'schen.
(*Author.*) 8vo. Berlin, 1912

Berlin, Preussische Akademie der Wissenschaften:
Geschichte des Fixsternhimmels.
(*Purchased.*) 4to. Karlsruhe, 1924
Abteilung I., Band 3. 2^h Rektaszension.

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Communication sur la " Méthode des Secteurs " en triangulation.
(*Service topographique fédéral.*) 8vo. [Berne], 1925

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Histoire de l'astronomie.
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——: Carte photographique du Ciel. Zones $+12°$, $+14°$ (20 Charts).
(*French Minister of Public Instruction.*)
——: Second Catalogue de l'Observatoire de Bordeaux, publié sous la direction de M. L. Picart. Zones $+10°$ à $+18°$.
(*Abbadia Observatory.*) 4to. Hendaye, 1924

Bordeaux, Société Astronomique:
L'Information astronomique. 1^{re} Année, Nos. 1, 2.
(*Society.*) Fol. sheet. [Bordeaux], 1924

Boston, American Academy of Arts and Sciences:
Memoirs. Vol. 14, No. 5.
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(*Editor.*) 4to. Hauniae, 1924

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Report of the Meeting, 1924.
(*Purchased.*) 8vo. London, 1925

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The Riddle of the Universe.
(*Author.*) 8vo. [Privately printed], 1924

Budapest, Hungarian State Observatory:
" Stella " Csillagaszati Egyešulet Almanachja 1925.
(*Observatory.*) Small 8vo. Budapest, 1924

Burnell (Arthur T.):
Moon Tables.
(*Author.*) 8vo. [Privately printed, *n.d.*]

Carmona (M. S. de):
La Organización Cometaria del Universo.
(*Author.*) 8vo. [Mexico, 1924]

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One Hundred Years of British Glass Making, 1824–1924.
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Terrestrial Magnetism. British (Terra Nova) Antarctic Expedition, 1910–1913.
(*Captain Scott Antarctic Fund*). 4to. London, 1921

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Modern Astrophysics.
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——: *Premier Congrès de l'Union Astronomique Internationale tenu à Rome du 2 au 10 Mai 1922.
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*Sur la structure de l'atmosphère solaire basse dite chromosphère.
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Elger (T. G.):
The Moon. A full description and map of its principal physical features.
(*Purchased.*) 8vo. London, 1895

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Band 6 (1 B), Heft 5.
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*Provisional report of the work of the International Latitude Service in the North Parallel +39° 8′ during the period September 6, 1922–March 6, 1924.
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Sternkunde und Sterndienst in Babel.
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Scientific Papers mainly on Electrodynamics and Natural Radiation.
(*Cambridge University Press.*)
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Epitome astronomiæ, qua brevi explicatione omnia, tam ad sphæricam quàm theoricam ejus partem pertinentia, ex ipsius scientiæ fontibus deducta, perspicuè per quæstiones traduntur.
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*Dispersione Anomala di Vapori "Triple Regolari."
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Histoire de la Longitude à la Mer au XVIII^e^. Siècle en France.
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Notes sur la Variabilité des Climats. Première partie. Introduction générale historique.
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Time and Time-keepers. Including the History, Construction, Care, and Accuracy of Clocks and Watches.
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Sir Isaac Newton's Mathematick Philosophy more easily demonstrated with Dr. Halley's account of Comets illustrated, being Forty Lectures read in the Publick Schools at Cambridge by William Whiston.
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*Zur Döllen'schen Methode der Zeitbestimmung.
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Observations of the Total Solar Eclipse of January 24, 1925:
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Tome 32, fasc. 2. M. Hamy, Le Spectrographe Stellaire de l'Observatoire de Paris.
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(*Observatory.*) 4to. Kiel, 1924

No. 5.* A. Prey, Über das System 70 Ophiuchi.

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